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Advanced Optics and Lasers Notes

FRANCESCO FAILLACE

PROF. ALESSIO GAMBETTA

1. EM WAVES BASICS REVIEW

From Maxwell's equations to wave equation

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \nabla \wedge \vec{E} = -\dfrac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{B} = 0 \\ \nabla \wedge \vec{H} = \dfrac{\partial \vec{D}}{\partial t} + \vec{J}_{FREE} \end{cases}$$

where $\vec{H} = \dfrac{\vec{B}}{\mu}$ and $\vec{D} = \varepsilon \vec{E}$.
Calling N the number of dipoles taken into account and $\vec{d}$ the displacement we can define the polarization vector as $\vec{P} = N \cdot \langle q \vec{d} \rangle = \varepsilon_0 \chi \vec{E}$, where χ is the susceptibility.
Assuming no free charges but only polarization charges it can be proved that

$$\rho_{Pol} = -\nabla \cdot \vec{P} = -\nabla \cdot (\varepsilon_0 \chi \vec{E}) \Rightarrow \nabla \cdot (\varepsilon \vec{E}) = -\nabla \cdot (\varepsilon_0 \chi \vec{E}) = \rho_{Pol}.$$

Hence, $\nabla \cdot [\varepsilon_0 (1+\chi) \vec{E}] = 0$. We call $\vec{D} = \varepsilon_0 (1+\chi) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$,

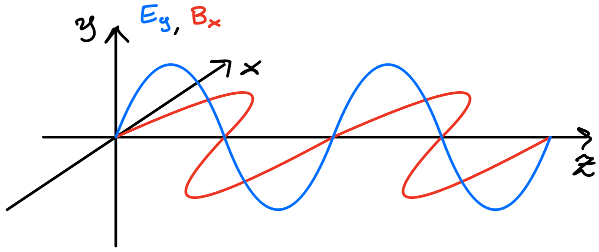
where $\varepsilon_r = 1 + \chi$ is the relative permittivity.
We call $\vec{D}$ displacement vector. In this way, in absence of free charges, we can write $\nabla \cdot \vec{D} = 0$. We also call $n = \sqrt{\varepsilon_r}$ refractive index.

Let us see how to retrive the wave equation from the Maxwell's ones. Consider the equations in free space and let's apply a curl to the second equation:

$$\nabla \wedge [\nabla \wedge \vec{E}] = \nabla \wedge \left[-\frac{\partial \vec{B}}{\partial t} \right] \Rightarrow \underbrace{\nabla (\nabla \cdot \vec{E})}_{=0} - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} (\nabla \wedge \vec{B})$$

$$\Rightarrow \nabla^2 \vec{E} = \frac{\partial}{\partial t} \left[\mu \varepsilon \frac{\partial \vec{E}}{\partial t} \right] \Rightarrow \nabla^2 \vec{E} - \mu \varepsilon \frac{\partial^2}{\partial t} \vec{E} = 0$$

Let us assume that the wave is a planar wave propagating along the z direction



$$\frac{\partial^2 \vec{E}}{\partial z^2} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Let us also assume that the electric field is polarized along the y axis. In this way the wave equation becomes

$$\frac{\partial^2 E_y}{\partial z^2} - \mu\varepsilon \frac{\partial^2 E_y}{\partial t^2} = 0 \quad \text{(linear polarization)}$$

The general integral of the former equation is given by:

$$\begin{cases} \vec{E}(z,t) = E_y \cos(kz - \omega t)\hat{u}_y \\ \vec{B}(z,t) = B_x \cos(kz - \omega t)\hat{u}_x \quad \text{(solution for the } \vec{B} \text{ eqn)} \end{cases}$$

where E_y and B_x are given by the boundary conditions.

Substituting the solution into the equation we obtain

$$\frac{\partial^2}{\partial z^2}\left[E_y \cos(kz - \omega t)\right] - \varepsilon\mu \frac{\partial^2}{\partial t^2}\left[E_y \cos(kz - \omega t)\right] = 0 \Rightarrow$$

$$\cancel{E_y}\left[-k^2 \cancel{\cos(kz - \omega t)}\right] - \varepsilon\mu \cancel{E_y}\left[-\omega^2 \cancel{\cos(kz - \omega t)}\right] = 0 \Rightarrow$$

$$k^2 + \varepsilon\mu\omega^2 = 0 \Rightarrow \left(\frac{k}{\omega}\right)^2 = \varepsilon\mu \Rightarrow \frac{k}{\omega} = \sqrt{\varepsilon\mu}, \quad \text{where } \omega = 2\pi\nu$$

and $k = \frac{2\pi}{\lambda}$. Hence $\frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}} \Rightarrow \frac{2\pi\nu}{2\pi/\lambda} = \lambda\nu = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \cdot \frac{1}{\sqrt{\varepsilon_r\mu_r}}$

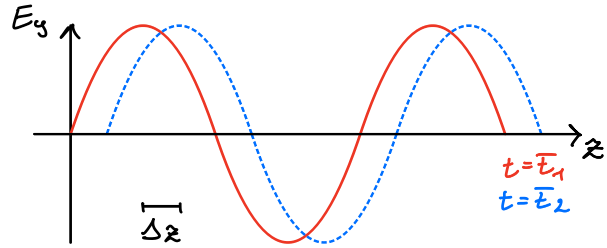
In non-ferromagnetic materials $\mu_r = 0 \Rightarrow \lambda\nu = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \cdot \frac{1}{n}$

Let's take for granted that $\frac{1}{\sqrt{\mu_0\varepsilon_0}} = c_0$. So, $\lambda\nu = \frac{c_0}{n} = c_n$

where c_n is the speed of light inside the medium.

Let us consider the former solution for the electric field

$\vec{E}(z,t) = E_y \cos(kz - \omega t)\hat{u}_y$. $\varphi = kz - \omega t$



The average speed of the wave is $\langle v \rangle = \dfrac{\Delta z}{\Delta t}$.

$$d\varphi(z,t) = \frac{\partial \varphi}{\partial z} dz + \frac{\partial \varphi}{\partial t} dt = 0 \Rightarrow k\,dz - \omega\,dt = 0 \Rightarrow \frac{dz}{dt} = \frac{\omega}{k}$$

where $\dfrac{dz}{dt} = v_{ph}$ (phase velocity). But $v_{ph} = \dfrac{\omega}{k} = \dfrac{1}{\sqrt{\mu\varepsilon}} = c_m$

$\dfrac{\omega}{k} = \lambda\nu = c_m$

Phasor notation

Let us consider the following electric field for a plane wave

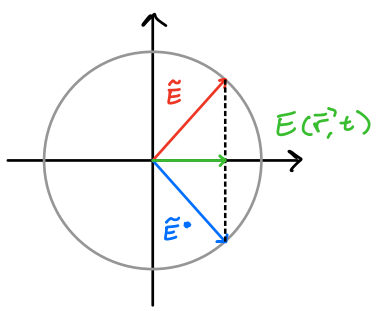
$E(\vec{r},t) = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t)$.

The phase of the wave is given by $\varphi(\vec{r},t) = \vec{k}\cdot\vec{r} - \omega t = \varphi_r - \omega t$

So, we can write

$$E(\vec{r},t) = E_0 \cos(\varphi_r - \omega t) = \mathrm{Re}\left\{E_0\, e^{j(\varphi_r - \omega t)}\right\} =$$

$$= \frac{1}{2}\left[E_0\, e^{j(\varphi_r - \omega t)} + E_0\, e^{-j(\varphi_r - \omega t)}\right] = \frac{1}{2}\left[\tilde{E}(\vec{r},t) + \tilde{E}^*(\vec{r},t)\right]$$



We can also rewrite

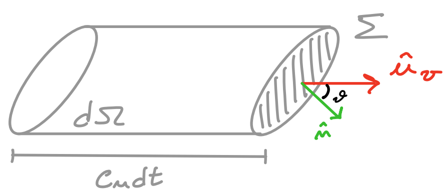
$$\tilde{E}_0(\vec{r}) = E_0\, e^{j\varphi_r} \Rightarrow \tilde{E}(\vec{r},t) = \tilde{E}_0(\vec{r})\, e^{-j\omega t}$$

Intensity of an EM wave

The intensity of a wave is given by the ratio between the power of the wave and the surface crossed by the wave.
In case of lasers we usually consider the surface as orthogonal to the beam

$$I = \frac{P}{\Sigma}, \quad \text{where} \quad P = \frac{dU}{dt} \quad \text{and} \quad U(t) = \frac{1}{2}\varepsilon E^2 + \frac{1}{2}\frac{B^2}{\mu}$$

$$B(t) = \frac{E(t)}{c_m} = \sqrt{\varepsilon\mu}\, E(t) \implies U(t) = \varepsilon E^2(t)$$



$$d\Omega = c_m dt \Sigma \hat{u}_v \cdot \hat{n} \implies$$

$$dU = U(t)\, d\Omega = \varepsilon E^2(t) c_m \, dt \, \Sigma \hat{u}_v \cdot \hat{n}$$

$$P = \frac{dU}{dt} = \varepsilon E^2(t) c_m \Sigma \hat{u}_v \cdot \hat{n}$$

So, the intensity accross Σ is given by

$$I_\Sigma = \frac{P}{\Sigma} = \varepsilon E^2(t) c_m \hat{u}_v \cdot \hat{n} = \varepsilon_o E^2(t) c_m \cos(\vartheta)$$

For a laser with $\hat{u}_v \parallel \hat{n}$ $\cos(\vartheta) = 1 \implies I(t) = \varepsilon c_m E^2(t)$

We will usually measure the average intensity

$$I = \langle I(t) \rangle = \varepsilon c_m \langle E^2(t) \rangle = \langle \vec{S}(t) \cdot \hat{n} \rangle$$

Let us rewrite the intensity in phasor notation

$$I = \varepsilon c_m \langle E^2(t) \rangle. \qquad E(t) = \frac{1}{2}\left(\tilde{E}(\vec{r},t) + \tilde{E}^*(\vec{r},t)\right) \implies$$

$$I = \varepsilon c_m \left\langle \frac{1}{4}\left(\tilde{E}^2(\vec{r},t) + \tilde{E}^{*2}(\vec{r},t) + 2\tilde{E}(\vec{r},t)\tilde{E}^*(\vec{r},t)\right)\right\rangle$$

An electromagnetic wave is an ergodic process, hence:

$$I = \varepsilon c_m \cdot \frac{1}{4} \lim_{T \to +\infty} \left[\int_{-T}^{T} \left(\overbrace{\tilde{E}^2(\vec{r}, t)}^{=0} + \overbrace{\tilde{E}^{*2}(\vec{r}, t)}^{=0} + 2\,\tilde{E}(\vec{r}, t)\,\tilde{E}^*(\vec{r}, t) \right) dt \right]$$

$$2\,\tilde{E}(\vec{r}, t)\,\tilde{E}^*(\vec{r}, t) = 2\,|\tilde{E}(\vec{r}, t)|^2$$

$$\lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{+T} E_0^2\, dt = \lim_{T \to +\infty} \frac{1}{\cancel{2T}}\, 2 E_0^2 \cdot \cancel{2T} = 2 E_0 \implies$$

$$I = \varepsilon c_m \cdot \frac{1}{4} \cdot 2 E_0^2 = \frac{1}{2} \varepsilon c_m E_0^2 \implies I = \frac{1}{2} \varepsilon_0 c_0 n E_0^2$$

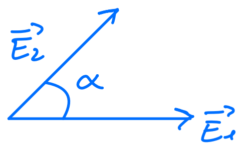
2. INTERFERENCE AND INTERFEROMETERS

Interference and beatnotes

Let us consider two laser beams with electric fields $\tilde{E}_1$ and $\tilde{E}_2$. The intensity on a screen is given by $I = I_1 + I_2 + I_{int}$. To observe interference we must have $\vec{E}_1 \parallel \vec{E}_2$, otherwise

$$\vec{E}(\vec{r},t) = \vec{E}_1(\vec{r},t) + \vec{E}_2(\vec{r},t), \quad \text{with} \quad \begin{cases} \vec{E}_1 = \vec{E}_{0,1}\cos(\varphi_1 - \omega_1 t) \\ \vec{E}_2 = \vec{E}_{0,2}\cos(\varphi_2 - \omega_2 t) \end{cases}$$

The intensity is $I = \varepsilon_0 c_0 n \langle E^2(\vec{r},t)\rangle$, where



$$E^2(\vec{r},t) = \vec{E}\cdot\vec{E} = (\vec{E}_1 + \vec{E}_2)\cdot(\vec{E}_1 + \vec{E}_2) =$$

$$= E_1^2 + E_2^2 + \vec{E}_1\cdot\vec{E}_2 + \vec{E}_2\cdot\vec{E}_1 = E_1^2 + E_2^2 + 2E_1E_2\cos\alpha$$

$$\Rightarrow I = \varepsilon_0 c_0 n\langle E_1^2\rangle + \varepsilon_0 c_0 n\langle E_2^2\rangle + 2\varepsilon_0 c_0 n\langle E_1E_2\rangle\cos(\alpha)$$

Hence, $I_{int} = 0$ if $\alpha = \frac{\pi}{2}$. Let's compute it using the phasor notation, for now assuming $\vec{E}_1 \parallel \vec{E}_2$

$$\tilde{E} = \tilde{E}(\vec{r},t) = \tilde{E}_1 + \tilde{E}_2$$

$$I = \frac{1}{2}\varepsilon_0 c_0 n E_0^2 = \frac{1}{2}\varepsilon_0 c_0 n|\tilde{E}|^2 = \frac{1}{2}\varepsilon_0 c_0 n\tilde{E}\cdot\tilde{E}^* \Rightarrow$$

$$I = \frac{1}{2}\varepsilon_0 c_0 n(\tilde{E}_1 + \tilde{E}_2)(\tilde{E}_1^* + \tilde{E}_2^*) = \frac{1}{2}\varepsilon_0 c_0 n\left(|\tilde{E}_1|^2 + |\tilde{E}_2|^2 + \tilde{E}_1\tilde{E}_2^* + \tilde{E}_2\tilde{E}_1^*\right)$$

Note that $\tilde{E}_2\cdot\tilde{E}_1^* = (\tilde{E}_1\tilde{E}_2^*)^*$ and $|\tilde{E}_1| = E_{01}$ $\Rightarrow$

$$I = \frac{1}{2}\varepsilon_0 c_0 n E_{01}^2 + \frac{1}{2}\varepsilon_0 c_0 n E_{02}^2 + \frac{1}{\cancel{2}}\varepsilon_0 c_0 n\,\cancel{2}\,\mathrm{Re}\{\tilde{E}_1\tilde{E}_2^*\} \Rightarrow$$

$$I = I_1 + I_2 + \varepsilon_0 c_0 n \,\mathrm{Re}\left\{\tilde{E}_1 \cdot \tilde{E}_2^*\right\} \Rightarrow$$

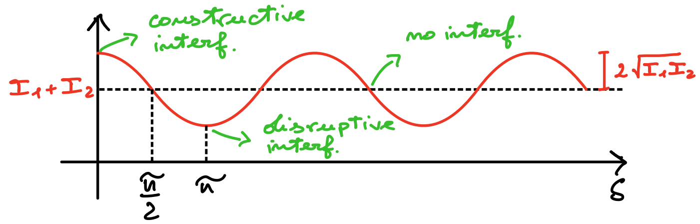
$$I = I_1 + I_2 + \varepsilon_0 c_0 n E_{01} E_{02} \,\mathrm{Re}\left\{ e^{j[\overbrace{(\varphi_1-\varphi_2)}^{\delta} - \overbrace{(\omega_1-\omega_2)}^{\Delta\omega}]t} \right\} \Rightarrow$$

$$I = I_1 + I_2 + \underbrace{\varepsilon_0 c_0 n E_{01} E_{02}}_{2\sqrt{I_1 I_2}} \cos(\delta - \Delta\omega \cdot t) \Rightarrow$$

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\delta - \Delta\omega \cdot t)$$

Observation. If $\Delta\omega = 0$ $\quad I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\delta)$.

In this case interference is only due to spatial phase and it is independent on time.



We define the visibility as: $V = \dfrac{I_{MAX} - I_{MIN}}{I_{MAX} + I_{MIN}}$

Since $I_{MAX} = I_1 + I_2 + 2\sqrt{I_1 I_2}$ and $I_{MIN} = I_1 + I_2 - 2\sqrt{I_1 I_2}$

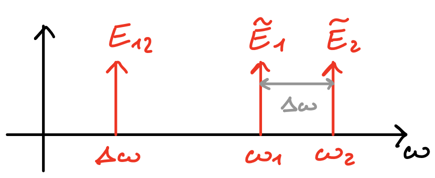
$$V = \frac{I_1 + I_2 + 2\sqrt{I_1 I_2} - (I_1 + I_2 - 2\sqrt{I_1 I_2})}{I_1 + I_2 + 2\sqrt{I_1 I_2} + I_1 + I_2 - 2\sqrt{I_1 I_2}} = 2\frac{\sqrt{I_1 I_2}}{I_1 + I_2}$$

If $I_1 = I_2 = I_0$, $\quad V = 2\dfrac{\sqrt{I_0^2}}{2 I_0} = 1$ and $I = 2 I_0 (1 + \cos\delta)$

Let us now consider $\delta = 0$ and $\Delta\omega \neq 0$ (beatnote condition)

In this case $I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\Delta\omega t)$.

$V = \dfrac{I_{MAX} - I_{MIN}}{I_{MAX} + I_{MIN}}$. $\quad$ If $I_1 = I_2 = I_0$, $\quad V = V_{MAX} = 1$



In the optical range $\nu = \dfrac{\omega}{2\pi} \in [10-500]$ THz,

but $\omega_2 - \omega_1$ can fall in the RF range

The $\Delta\omega$ component is called beatnote.

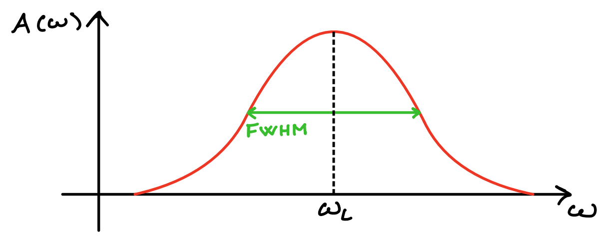
Wave packet solution of the wave equation

Let us study the wave packet solution of the wave equation $\tilde{E}(z,t) = A e^{j(\beta z - \omega t)}$.

Since the wave equation is linear, it satisfies the superposition principle. Hence, a solution of the equation is

$$\tilde{E}(z,t) = A_1 e^{j(\beta_1 z - \omega_1 t)} + A_2 e^{j(\beta_2 z - \omega_2 t)}, \quad \text{with } \beta = \frac{2\pi}{\lambda_m} = \frac{2\pi}{\lambda_0} n$$

Passing to a continuum of frequencies we can write the field as the Fourier transform of the spectrum



$$\tilde{E}(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}(\omega - \omega_L)\, e^{j(\beta z - \omega t)}\, d\omega$$

Note that we work with the baseband amplitude spectrum into the integral.

The propagation vector depends on the wavelength, hence it depends on the frequency. Linearizing it we obtain

$$\beta = \beta(\omega) \simeq \beta_L + \frac{d\beta}{d\omega}(\omega - \omega_L) \quad \text{(first order dispersion). So,}$$

$$\tilde{E}(z,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(\omega - \omega_L) \exp\left[j\left(\left(\beta_L + \frac{d\beta}{d\omega}(\omega - \omega_L) \right) z - \omega t \right) \right] d\omega \cdot e^{-j\omega_L} e^{j\omega_L}$$

$$= \frac{e^{-j\omega_L t}}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}(\omega - \omega_L) \exp\left[j\left(\beta_L z + \frac{d\beta}{d\omega}(\omega - \omega_L) z - (\omega - \omega_L) t \right) \right] \Rightarrow$$

$$\tilde{E}(z,t) = \frac{1}{2\pi} e^{j(\beta z - \omega t)} \int_{-\infty}^{+\infty} \tilde{A}(\Delta\omega)\, e^{j(\dot{\beta} \cdot \Delta\omega \cdot z - \Delta\omega \cdot t)}\, d(\Delta\omega) =$$

$$= \frac{1}{2\pi} e^{j(\beta z - \omega t)} \int_{-\infty}^{+\infty} \tilde{A}(\Delta\omega)\, e^{j(\dot{\beta} z - t)\Delta\omega}\, d(\Delta\omega), \quad \text{where}$$

$\dot{\beta} = \frac{d\beta}{d\omega} = \frac{1}{v_g}$ and v_g is the group velocity of the wave packet.

Hence, $\dot{\beta} z - t = \frac{z}{v_g} - t = \tau$. Thus, we can rewrite

$$\tilde{E}(z,t) = \frac{1}{2\pi} e^{j(\beta z - \omega t)} \int_{-\infty}^{+\infty} \tilde{A}(\Delta\omega)\, e^{j\Delta\omega\tau}\, d(\Delta\omega).$$ Note that

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{A}(\Delta\omega)\, e^{j\Delta\omega\tau}\, d(\Delta\omega) = \mathcal{F}^{-1}\left\{\tilde{A}(\Delta\omega)\right\}\Big|_{t=\tau} \equiv G(\tau)$$

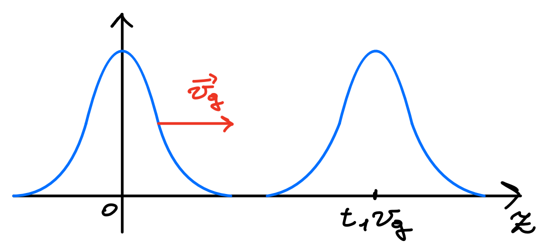
We call $G(\tau)$ envelope of the wave. The E field can be written

as: $$\tilde{E}(z,t) = \underbrace{e^{j(\beta_L z - \omega_L t)}}_{\text{Carrier}} \underbrace{G(\tau)}_{\text{Envelope}}$$

The former expression is called wavepacket solution

The envelope moves in space with the speed of v_g. Indeed

we know that $G(\tau) = G\left(\frac{z}{v_g} - t\right)$



In $\tau = 0$, for $t = 0$ $\quad \frac{z}{v_g} - t = 0 \Rightarrow z = 0.$

For $t = t_1$ $\quad \frac{z}{v_g} - t_1 = 0 \Rightarrow z = t_1 v_g$

Let's take the differential of τ $\quad d\tau = d\left(\frac{z}{v_g} - t\right)$. For $\tau = 0$

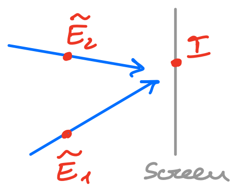
$\frac{dz}{v_g} - dt = 0 \Rightarrow \frac{dz}{dt} = v_g$. This proves that, for a fixed

τ, the wave packets moves with velocity $v_g = \left(\frac{d\beta}{dt}\right)^{-1}$

Since $G(\tau)$ is related to $\tilde{A}(\Delta\omega)$ by a Fourier transform relationship, calling $\Delta\tau_{FWHM}$ the temporal width of the pulse and $\Delta\omega_{FWHM}$ the spectral width of the spectrum, we must have $\Delta\omega_{FWHM} \cdot \Delta\tau_{FWHM} \geq 2\pi$. This relation is the uncertainty principle of the Fourier transform.

<u>Interferometers</u>

We saw that in order to have interference we need at least two waves. Let us consider two harmonic plane waves that are interacting on a point on a screen



$$I = I_1 + I_2 + \sqrt{I_1 I_2}\,\cos(\delta + \Delta\omega t)$$

The beatnote term $\Delta\omega t$ will be at very low frequency, so it is no visible.

This is the reason why we do not consider it as part of the interference and we will always write

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\cos(\delta).$$

The requirements to have interference are:

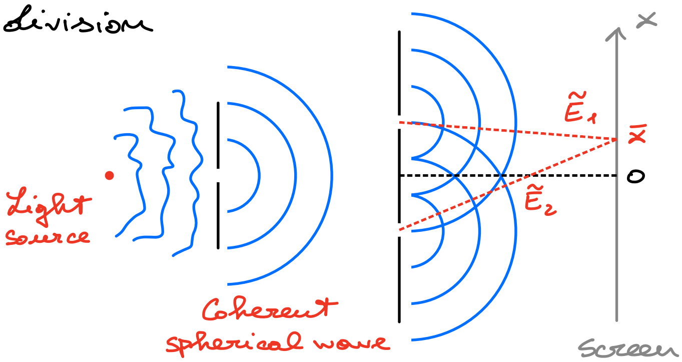
1) $\omega_1 = \omega_2$.
2) $\vec{\tilde{E}}_1 \parallel \vec{\tilde{E}}_2$ (parallel polarization).
3) They must have some amount of mutual coherence, i.e. $\delta = (\vec{k}_1 \cdot \vec{r} - \vec{k}_2 \cdot \vec{r})$ is constant in time.

To satisfy 1 and 3 we should use a single source.

We have to split the source in to paths using wavefront division interferometry or amplitude division interferometry.

Young interferometer

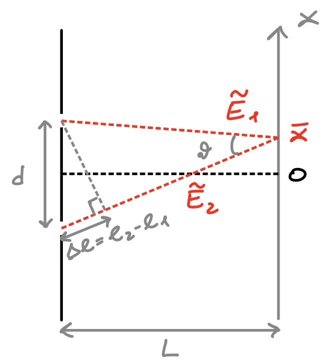
The Young interferometer is an example of wavefront division



$$\begin{cases} \tilde{E}_1 = A\, e^{(jk\ell_1 + \varphi_0)} \\ \tilde{E}_2 = A\, e^{(jk\ell_2 + \varphi_0)} \end{cases}$$

we call $k\ell + \varphi_0 = \varphi_L$ and

$$\delta = \varphi_2 - \varphi_1 = k(\ell_2 - \ell_1)$$



$\Delta\ell = d \sin\vartheta$. $\Delta\ell$ is called optical path difference. In this way we can write

$$\delta = k\Delta\ell = \frac{2\pi}{\lambda} d \sin(\vartheta).$$

If ϑ is very small $\sin\vartheta \simeq \tan\vartheta = \frac{x}{L}$, so $\delta(\vartheta) = \delta(x) = \frac{2\pi}{\lambda} d \frac{x}{L}$

$I(x) = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\delta)$. If $I_1 = I_2 = I_0$,

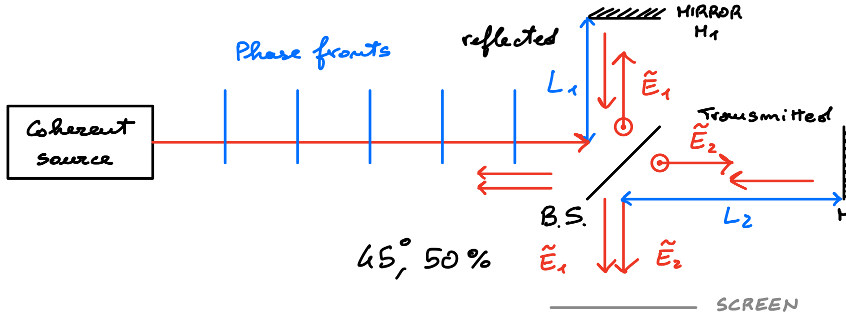
$$I(x) = 2I_0 + 2I_0 \cos\left(\frac{2\pi}{\lambda}\frac{d}{L}x\right) = 2I_0\left[1 + \cos\left(\frac{2\pi}{\lambda}\frac{d}{L}x\right)\right].$$

So, if $x = 0$ $I_0 = 4I_0$ and $I = 0$ if $\frac{\cancel{2\pi}}{\lambda}\frac{d}{L}x = \cancel{2}m\cancel{\pi}$

$$\Rightarrow x = \frac{\lambda L}{d} m$$

Michelson interferometer

Let us consider a coherent source of plane waves and assume to use the setup in the figure below, called Michelson interferometer.



We call

$d \equiv L_2 - L_1$

The phase accumulated by the two beams is given by

$$\begin{cases} \varphi_1 = \varphi_0 + 2kL_1 \\ \varphi_2 = \varphi_0 + 2kL_2 \end{cases} \Rightarrow \delta = \varphi_2 - \varphi_1 = 2k(L_2 - L_1) = \frac{4\pi}{\lambda} \cdot d$$

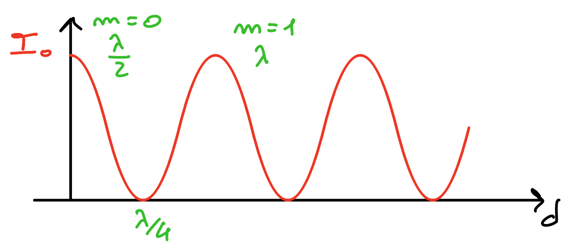
The intensity pattern that we observe on the screen is

$$I_{screen} = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos(\delta). \qquad I_1 = I_2 = \frac{I_0}{4}, \text{ hence}$$

$$I_{screen} = \frac{I_0}{2} + 2\left(\frac{I_0}{4}\right)\cos\delta = \frac{I_0}{2}\left[1 + \cos\left(\frac{4\pi}{\lambda}d\right)\right]$$

The intensity on the screen will be constant, but the whole intensity will change if we change d

$$I_{screen} = I_{MAX} \quad \text{if} \quad \cos\left(\frac{4\pi}{\lambda}d\right) = 1 \Rightarrow \frac{{}^{2}\cancel{4}\pi}{\lambda}d = \cancel{2}m\cancel{\pi} \Rightarrow d = m\frac{\lambda}{2}, \quad m \in \mathbb{N}$$



This interferometer converts a distance change into an intensity variation with sensibility of a fraction of λ

In the opposite way it can be exploited to convert a change in d into a ΔI intensity variation to find variations of λ

$$d = m\frac{\lambda}{2} \Rightarrow \lambda = \frac{2d}{m}. \quad \text{Measuring at } \bar{d}, \text{ we can measure } \lambda = \frac{2\bar{d}}{m}$$

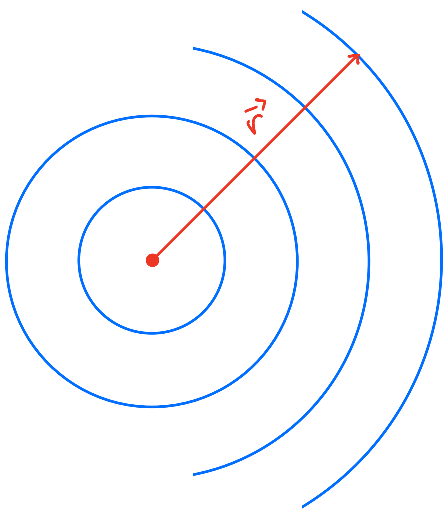
The issue is that it is very hard to precisely measure d

$$d_{MEAS} = \bar{d} + \underbrace{\varepsilon_d}_{error} \Rightarrow \lambda_{MEAS} = \frac{2\, d_{MEAS}}{\bar{m}} = \frac{2\bar{d}}{\bar{m}} + \frac{2\varepsilon_d}{\bar{m}} = \bar{\lambda} + \varepsilon_\lambda$$

where $\bar{\lambda}$ is the real wavelength and ε_λ the error. Note that

$\varepsilon_\lambda = \dfrac{2\varepsilon_d}{\bar{m}}$, so the error decreases increasing $\bar{m}$.

Let us now consider the illumination from a pointlike source; it generates spherical waves.



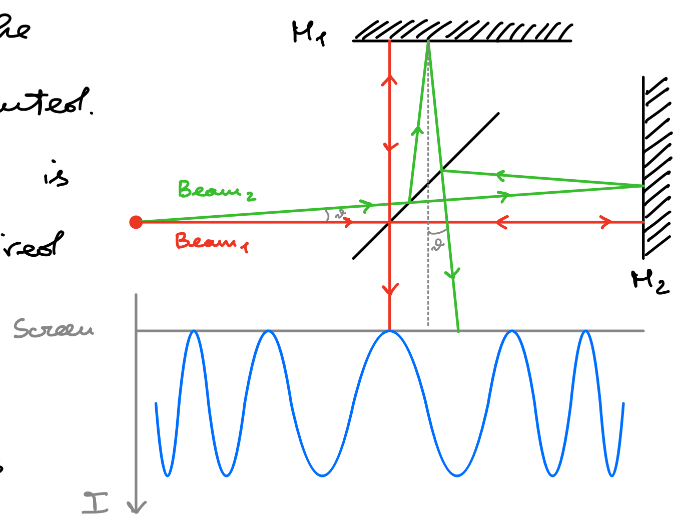
Phase is constant if r is constant, where $r = |\vec{r}|$. In this case we have always $\vec{r} \parallel \vec{k}$, so we can write

$$\tilde{E}_{spher} = A e^{j(kr - \omega t)}.$$

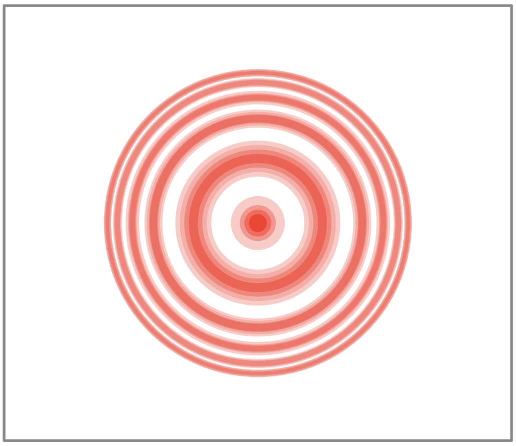
In this case we can imagine it to be a source of infinite plane waves, all with different sources.

In the figure aside two of the infinite directions are represented. The phase acquired by Beam 1 is different from the one acquired by Beam 2.

If Beam 1 is parallel to the optical axis and Beam 2 forms an angle ϑ with the axis we can prove that



$$\delta_1 = \frac{4\pi}{\lambda_1} d \quad \text{and} \quad \delta_2 = \frac{4\pi}{\lambda_2} d \cos(\vartheta),$$ giving the intensity pattern shown in the figure above.



What we see on the screen is a bright spot in the center and concentric circular fringes, as shown in the figure aside.

Counting the fringes varying the distance, we know the number of maxima $\bar{m}$. Knowing $\bar{d}$ we find the wavelength of the beam.

We already saw that with this interferometer we can measure wavelengths. With the same setup we can take a very precise measurement of the speed of light.

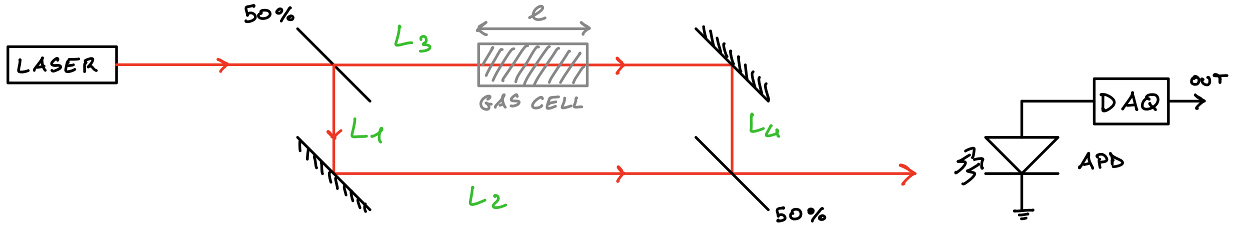
Let us assume to be in vacuum. In this case $\lambda_0 \nu = c_0$.

If ν is known with very high accuracy it is sufficient to measure λ_0 to find c_0.

Mach-Zender interferometer

This device exploits the same principle of the Michelson interferometer to measure lengths and phase shifts.

The setup is the following.



In general, if $(L_3+L_4)-(L_2+L_1)=0$ we say that the interference is balanced. Let us assume that the gas cell is filled with a well known refractive index n. In this way the optical path L_3+L_4 changes in:

$$(L_3+L_4)' = L_3 - \ell + n\ell + L_4 = L_3+L_4+(n-1)\ell.$$

So, the optical path difference is

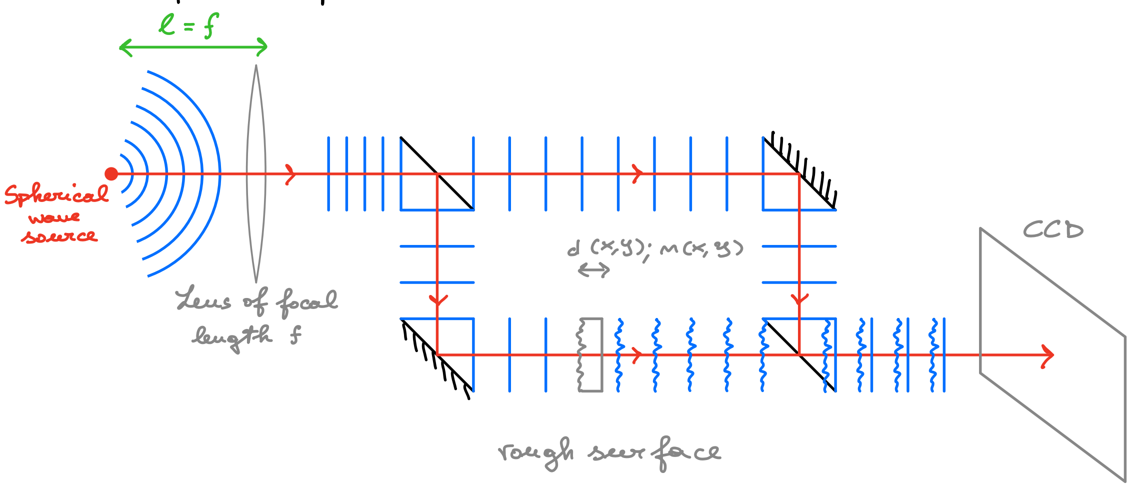
$$\delta = \left\{(L_1+L_2)+(L_3+L_4)' - \left[(L_1+L_2)+(L_3+L_4)\right]\right\}\cdot \frac{2\pi}{\lambda} \Rightarrow$$

$$\delta = \frac{2\pi}{\lambda}(n-1)\ell, \quad \text{so we find } \ell = \frac{\delta}{2\pi}\cdot\frac{\lambda}{n-1}.$$

In the same way, knowing the length of the cell, we can find the refractive index of an unknown transparent material: $n = 1 + \delta\dfrac{\lambda}{2\pi\ell}$

Twiman-Green configuration

This is a configuration of the Mach-Zender interferometer that exploits plane waves



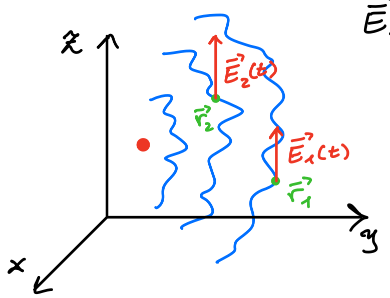
If we put an object with a rough surface (or with non-constant refractive index) we obtain

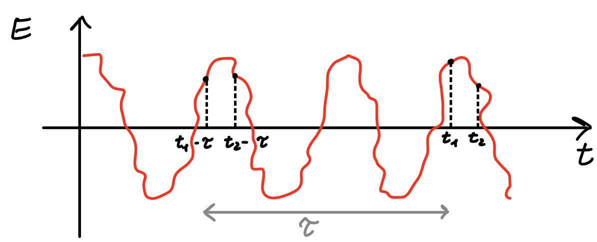
$$\delta(x, y) = \frac{2\pi}{\lambda_0} n \, d(x, y)$$

<u>Coherence</u>

We saw that to have interference between two waves we must have: same frequency, same polarization and some degree of coherence.

Let us consider a general source of EM waves and a field $\vec{E}_1(t)$ in a point $\vec{r}$ of the space





The degree of temporal coherence is defined as

$$\tilde{\gamma}(\tau) = \frac{\langle \tilde{E}_1^*(t) \cdot \tilde{E}_1(t-\tau) \rangle}{\sqrt{\langle |\tilde{E}_1(t)|^2 \rangle} \cdot \sqrt{\langle |\tilde{E}_1(t-\tau)|^2 \rangle}}$$

This is the complex degree of temporal coherence. The numerator is the complex autocorrelation function of the electric field:

$$\tilde{\Gamma}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \tilde{E}^*(t) \, \tilde{E}(t-\tau) \, dt$$

The real autocorrelation function is $\Gamma(\tau) = \mathrm{Re}\{\tilde{\Gamma}(\tau)\}$

The spatial coherence is defined by the correlation of $\vec{E}$ between two points in space $\vec{r}_1$, $\vec{r}_2$ at the same time. Assuming the two fields as parallel we obtain

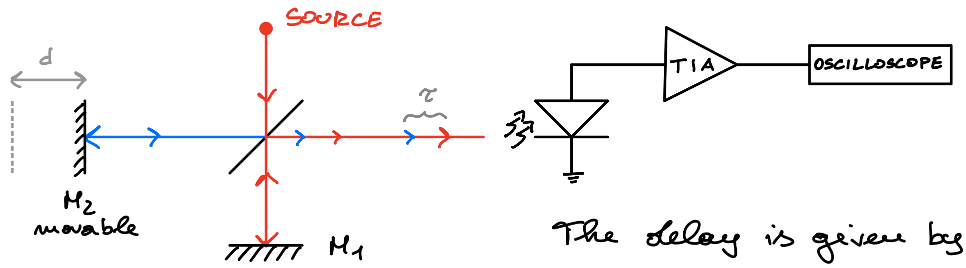
$$\tilde{\gamma}(\vec{r}_1, \vec{r}_2) = \frac{\langle \tilde{E}_1^*(t)\, \tilde{E}_2(t) \rangle}{\sqrt{\langle |E_1(t)|^2 \rangle}\sqrt{\langle |E_2(t)|^2 \rangle}} \, .$$

In general we can write the complex degree of mutual coherence

as $$\tilde{\gamma}(\vec{r}_1, \vec{r}_2, t) = \frac{\langle \tilde{E}_1^*(t) \cdot \tilde{E}_2(t-\tau) \rangle}{\sqrt{\langle |\tilde{E}_1(t)|^2 \rangle} \cdot \sqrt{\langle |\tilde{E}_2(t-\tau)|^2 \rangle}}$$

Having a high coherence means to have a good monochromatic source, that ideally means that the electric field oscillates at a single frequency ν_0.

Let us see how does temporal coherence affect the measurement with a Michelson interferometer



The delay is given by $\tau = \frac{2d}{c_0}$

The intensity that we detect with the photodiode is

$$I = \frac{1}{2}\varepsilon_0 c_0 n \langle |\tilde{E}_{TOT}(t,\tau)|^2 \rangle, \quad \text{where} \quad \begin{cases} \tilde{E}_{TOT} = \tilde{E}_1(t) + \tilde{E}_2(t) \\ \tilde{E}_2 = \tilde{E}_1(t-\tau) \end{cases}$$

$$I = \frac{1}{2}\varepsilon_0 c_0 n \langle [\tilde{E}_1(t) + \tilde{E}_1(t-\tau)] \cdot [\tilde{E}_1(t) + \tilde{E}_1(t-\tau)]^* \rangle =$$

$$= \frac{1}{2}\varepsilon_0 c_0 n \langle |\tilde{E}_1(t)|^2 \rangle + \frac{1}{2}\varepsilon_0 c_0 n \langle |E_1(t-\tau)|^2 \rangle +$$

$$+ \frac{1}{2}\varepsilon_0 c_0 n \cdot 2\,\mathrm{Re} \langle \tilde{E}_1^*(t)\, E_1(t-\tau) \rangle \implies$$

$$I = 2I_1 + \varepsilon_0 c_0 n\, \mathrm{Re}\{\langle \tilde{E}^*(t)\, \tilde{E}_1(t-\tau) \rangle\} = 2I_1 + \varepsilon_0 c_0 n\, \tilde{\Gamma}(\tau)$$

where I_0 is the intensity emitted from the light source coupled to the interferometer and $I_1 = \frac{I_0}{4}$.

We can rewrite

$$I = 2I_1 + \varepsilon_0 c_0 n \, \mathrm{Re}\left\{\langle |\tilde{E}_1(t)|^2 \rangle\right\} = 2I_1 + 2I_1 \mathrm{Re}\left\{\tilde{\gamma}(\tau)\right\} \Rightarrow$$

$$I = 2I_1\left[1 + \mathrm{Re}\left\{\tilde{\gamma}(\tau)\right\}\right] \underset{I_1 = \frac{I_0}{4}}{\Longrightarrow} I = \frac{1}{2} I_0 \left[1 + \mathrm{Re}\left\{\tilde{\gamma}(\tau)\right\}\right]$$

Remember that τ is related to the distance between the two mirrors.

Example: if $E(t) = E_0 \cos(\varphi + \omega_0 t)$ (monochromatic wave)

we can rewrite it as $\tilde{E}(t) = E_0 e^{j\varphi} e^{-j\omega_0 t} \Rightarrow$

$$\tilde{\gamma}(\tau) = \frac{\langle \tilde{E}^*(t) \cdot \tilde{E}(t-\tau) \rangle}{\sqrt{\langle |\tilde{E}(t)|^2 \rangle} \cdot \sqrt{\langle |\tilde{E}(t-\tau)|^2 \rangle}} = \frac{\langle \tilde{E}^*(t) \cdot \tilde{E}(t-\tau) \rangle}{E_0^2} =$$

$$= \frac{1}{E_0^2} \langle E_0 \cancel{e^{-j\varphi}} e^{j\omega_0 t} \cdot E_0 \cancel{e^{j\varphi}} e^{-j\omega_0 (t-\tau)} \rangle =$$

$$= \frac{1}{E_0^2} \langle E_0^2 \cancel{e^{j\omega_0 t}} \cancel{e^{-j\omega_0 t}} e^{j\omega_0 \tau} \rangle = \frac{E_0^2 e^{j\omega_0 \tau}}{E_0^2} \Rightarrow \tilde{\gamma}_{\text{Monochr.}}(\tau) = e^{j\omega_0 \tau}$$

Hence, in case of monochromatic wave, the moving mirror causes a phase change:

$$|\tilde{\gamma}(\tau)| = 1; \quad \measuredangle\left[\tilde{\gamma}(\tau)\right] = \omega_0 \tau; \quad \mathrm{Re}\left\{\gamma(\tau)\right\} = \cos(\omega_0 \tau)$$

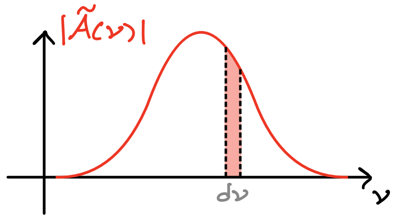
$$I_{\text{MICH.}} = \frac{I_0}{2}\left[1 + \mathrm{Re}\left\{\tilde{\gamma}(\tau)\right\}\right] = \frac{I_0}{2}\left[1 + \cos(\omega_0 \tau)\right]$$

We can rewrite $\omega_0 \tau$ as: $\omega_0 \tau = \frac{2\pi c_0}{\lambda_0} \cdot \frac{2d}{c_0} \Rightarrow \omega_0 \tau = \frac{4\pi d}{\lambda_0}$, so

$$I_{\text{MICH.}} = \frac{I_0}{2}\left[1 + \cos\left(\frac{4\pi}{\lambda_0} d\right)\right]$$

Power and absorption spectrum

We know from the Fourier transform theory that a single frequency component ν_0 corresponds to a sinusoid in time. Considering an

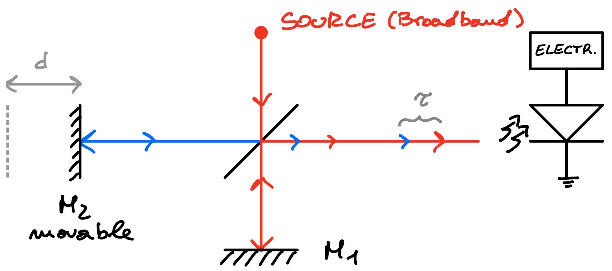


amplitude spectrum like the one in the figure aside we saw that we can write the electric field in time as

$$E(t) = A\, e^{j(\beta t - 2\pi\nu_0 t)} \cdot G\left(\frac{z}{2v_g} - t\right),$$ where $G(\tau)$ is the envelope.

We can define a power spectrum as $S(\nu) = |\tilde{A}(\nu)|^2$.

Let us consider a Michelson interferometer with a broadband light source.



$$I(\tau) = \frac{I_0}{2}\left[1 + \operatorname{Re}\{\tilde{\gamma}\}\right]$$

$$\tilde{\gamma}(\tau) = \frac{\langle \tilde{E}^*(t) \cdot \tilde{E}(t-\tau) \rangle}{\sqrt{\langle |\tilde{E}(t)|^2 \rangle} \cdot \sqrt{\langle |\tilde{E}(t-\tau)|^2 \rangle}}$$

$$\tau = \frac{2d}{c_0}$$

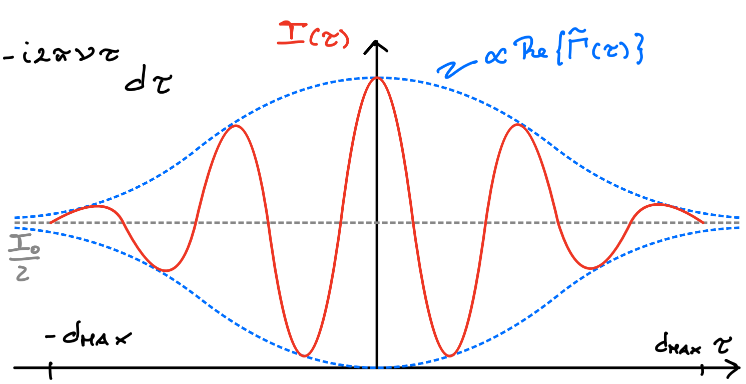
Note that the denominator of $\tilde{\gamma}$ is proportional to the intensity.

We can rewrite $\tilde{\gamma} = \dfrac{\tilde{\Gamma}(\tau)}{2I}\,\varepsilon_0 c_0 n \Rightarrow I(\tau) = \dfrac{I_0}{2} + \varepsilon_0 c_0 n \operatorname{Re}\{\tilde{\Gamma}(\tau)\}$

According to the Wiener-Khinchin we know that

$$S(\nu) = \mathcal{F}\{\tilde{\Gamma}(\tau)\} = \int_{-\infty}^{+\infty} \Gamma(\tau)\, e^{-i2\pi\nu\tau}\, d\tau$$

Graphically, the result of the interferometer as a function of τ is the following



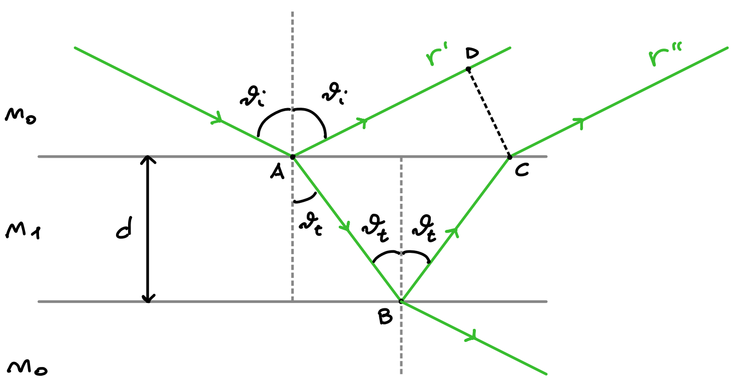
The former graphic is called interferogram.

To study the spectrum of a source it is sufficient to take the Fourier transform of the interferogram. This is very important for spectroscopy applications.

Multiple reflection through a thin plate

Let us study the interference due to double reflection through a thin plate.

$$r = \frac{\tilde{E}_r}{\tilde{E}_i} \; ; \qquad t = \frac{\tilde{E}_t}{\tilde{E}_i} .$$



Referring to the figure, r' and r'' can interfere constructively or destructively.

The optical path difference is given by

$$\Delta = n_0 \left(\overline{AB} + \overline{CD} \right) - n_1 \cdot \overline{AD}$$

$$\delta = \frac{2\pi}{\lambda_0} \Delta = \frac{2\pi}{\lambda_0} \left[n_1 \left(\overline{AB} + \overline{CD} \right) - n_0 \overline{AD} \right], \quad \text{where}$$

$$\overline{AB} = \frac{d}{\cos\vartheta_t} = \overline{BC} \implies \overline{AB} + \overline{BC} = 2\overline{AD} = \frac{2d}{\cos\vartheta_t} .$$

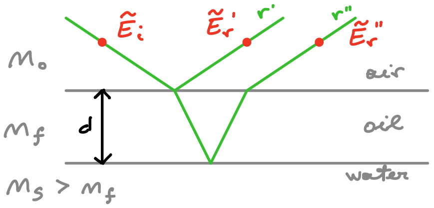
$$\overline{AD} = \overline{AC} \cdot \sin(\vartheta_i), \quad \text{where} \quad AC = 2\overline{AB} \sin(\vartheta_t)$$

$$\begin{cases} AD = 2\left[\dfrac{d}{\cos\vartheta_t} \sin(\vartheta_t) \right] \sin\vartheta_i \\ n_0 \sin(\vartheta_i) = n_1 \sin(\vartheta_t) \end{cases} \implies \overline{AD} = \frac{n_1}{n_0} \cdot \frac{2d}{\cos\vartheta_t} \sin^2\vartheta_t$$

So, $$\delta = \frac{2\pi}{\lambda}\left[n_1 \cdot \frac{2d}{\cos(\vartheta_t)} - \cancel{n_0} \cdot \frac{n_1}{\cancel{n_0}} \cdot \frac{2d}{\cos(\vartheta_t)} \sin^2(\vartheta_t)\right] \Rightarrow$$

$$\delta = \frac{2\pi}{\lambda}\left[n_1 \cdot \frac{2d}{\cos\vartheta_t} - n_1 \frac{2d}{\cos\vartheta_t} + n_1 \frac{2d}{\cos\vartheta_t}\cos^2\vartheta_t\right] = \frac{4\pi}{\lambda_0} n_1 d \cos(\vartheta_t)$$

Example: destructive interference from a thin layer deposited on a substrate. This is called Frizeau fringe effect (for



example when we have oil on water.

Let's see when $\delta = \pi$ (destructive).

Typically R_{film} and R_{sub} are small, of the order of $R = 4\%$, so $R_{plate} \simeq 0$

$\delta = \pi + \frac{4\pi}{\lambda_0} n_f \cos\vartheta_t = \pi + \frac{4\pi}{\lambda_0} n_f d$. We have destructive interference if $\frac{4\pi}{\lambda_0} n_f d = 2\pi \Rightarrow d = \frac{\lambda_0}{2 n_f} = \frac{\lambda}{2} \Rightarrow \lambda = \frac{\lambda_0}{n_f}$

Instead, we have constructive interference if $\delta_{TOT} = 2\pi$, so

$\frac{4\pi}{\lambda_0} n_f d = \pi \Rightarrow d = \frac{\lambda}{4}$

The two structures are called, respectively $\frac{\lambda}{2}$ and $\frac{\lambda}{4}$ waveplates.

Putting ourselves in the case of destructive interference we have

$$r_{TOT} = \frac{\tilde{E}_{r,TOT}}{\tilde{E}_N}, \quad \tilde{E}_{r,TOT} = \tilde{E}_r' + \tilde{E}_r''; \quad \tilde{E}_r' = r\tilde{E}_i, \quad \tilde{E}_r'' = t\, r' t' \tilde{E}_i e^{i\delta}$$

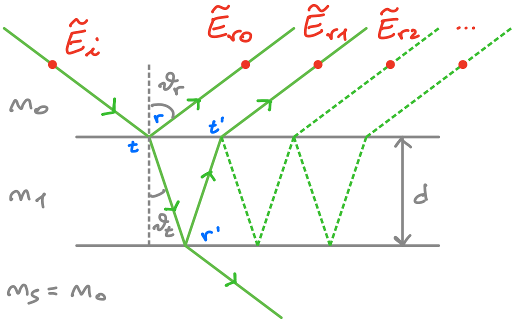
$tt' = T = 1 - R$. $R = 0 \Rightarrow tt' = 1$, so $\tilde{E}_r'' = -r'\tilde{E}_i$.

If $r = r'$, $r_{TOT} = \frac{r\tilde{E}_i - r'\tilde{E}_i}{\tilde{E}_i} = 0$. $r = r' \Rightarrow \left(\frac{n_0 - n_f}{n_0 + n_f}\right) = \left(\frac{n_f - n_s}{n_f + n_s}\right) \Rightarrow$

$$\frac{1 - \frac{n_f}{n_0}}{1 + \frac{n_f}{n_0}} = \frac{1 - \frac{n_s}{n_f}}{1 + \frac{n_s}{n_f}}.$$ $r = r'$ if $\frac{n_f}{n_0} = \frac{n_s}{n_f} \Rightarrow n_f^2 = n_s n_0 \Rightarrow n_f = \sqrt{n_s n_0}$

So, $n_f = \sqrt{n_s n_0}$ is the condition to have $r_{TOT} = 0 \Rightarrow R_{sys} = 0$

Let us now consider multiple interference considering a thin plate in air.



If $R_{n_0 n_1}$ is small we can consider only $\tilde{E}_{r0}$ and $\tilde{E}_{r1}$.

If not, let us look for the overall R and T of this system:

$$r_{overall} = \frac{\tilde{E}_{r0} + \tilde{E}_{r1} + \tilde{E}_{r2} + \dots}{\tilde{E}_i}$$

$\tilde{E}_{r0} = r\,\tilde{E}_{ri}$; $\tilde{E}_{r1} = t r' t' \tilde{E}_{ri} \cdot e^{i\delta}$, where $\delta = \frac{4\pi}{\lambda_0} n_1 d \cos\vartheta_t$

So, for the N_{th} E_r we have $E_{rN} = tt' \cdot (r')^{2N-1} E_{ri}\, e^{iN\delta}$, $N > 0$

Hence, $\tilde{E}_{rTOT} = \left\{ r + r'tt' e^{i\delta} \left[1 + \sum_{k=1}^{\infty} (r')^{2k} e^{ik\delta} \right] \right\} \tilde{E}_i \Rightarrow$

$$\tilde{E}_{r,TOT} = \left\{ r + r'tt' e^{i\delta} \sum_{k=0}^{\infty} \left[(r')^2 e^{i\delta} \right]^k \right\} \tilde{E}_i$$

The former series is a geometrical one, so we can write

$$\tilde{E}_{r,TOT} = r + \frac{r'tt'}{1-(r')^2 e^{i\delta}} e^{i\delta}$$

From the last relation we can also write that

$I_{r,TOT} \propto \tilde{E}_{r,TOT} \tilde{E}^*_{r,TOT}$ (and, of course, $I_i \propto \tilde{E}_i \tilde{E}_i^*$).

$$R_{TOT} = \frac{I_{r,TOT}}{I_i} = \frac{F \sin^2\left(\frac{\delta}{2}\right)}{1 + F \sin^2\left(\frac{\delta}{2}\right)} ; \quad \text{where} \quad \delta = \frac{4\pi}{\lambda_0} n_1 d \cos(\vartheta_t)$$

and $F = \frac{4 R_s}{(1-R_s)^2}$. F is called finesse or contrast factor.

The total trasmissivity of the system will be given by

$$T_{TOT} = 1 - R_{TOT} = \frac{1}{1 + F\sin^2\left(\frac{\delta}{2}\right)}$$

The relations of R_{TOT} and T_{TOT} are called Airy formulae.

The Fabry-Perot interferometer

The thin plate structure is called Fabry-Perot interferometer if $R_s \to 1$, so if F is high (conventionally we may say $F > 100$).

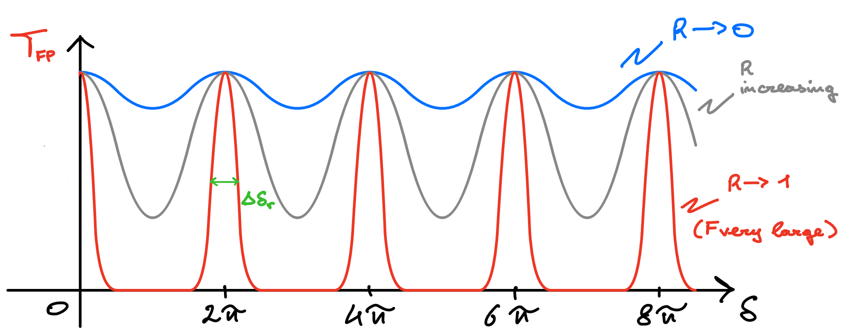
Let us study the trasmission T_{FP} as a function of δ:

$$T_{FP} = \frac{1}{1 + F\sin^2\left(\frac{\delta}{2}\right)};$$

$$\delta = \frac{4\pi}{\lambda_0} n_1 d \cos(\vartheta_t)$$

We can see that

$T_{FP,MAX} = 1$ if $\delta = 2\pi m$



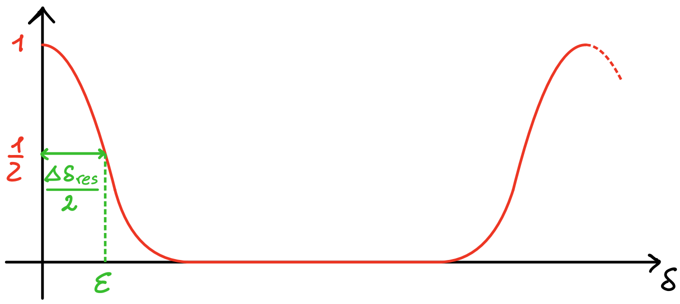
If $F \to 0$ ($R_s \simeq 0$) we have that $T = 1 - F\sin^2\left(\frac{\delta}{2}\right)$ (in blue)

In this case $T_{P,MIN} = 1 - F$ (for $\delta = \pi + 2m\pi$). We can lower this minimum value by increasing the reflectivity.

For $R \to 1 \Rightarrow F \to \infty$, we obtain the plot in red in the figure above. The sharper are the lines in the phase domain, the higher is the resolution of the F.P. interferometer, that can be used in spectroscopy applications.

We define $\Delta\delta_{res}$ = FWHM of each peak, so:

$\Delta\delta_{res} = 2\varepsilon$. We have to impose



$$\frac{1}{1+F\sin^2\left(\frac{\varepsilon}{2}\right)} = \frac{1}{2} \implies$$

$$F\sin^2\left(\frac{\varepsilon}{2}\right) = 1$$

If ε is very small $\sin^2\left(\frac{\varepsilon}{2}\right) = \left(\frac{\varepsilon}{2}\right)^2 \implies F\sin^2\left(\frac{\varepsilon}{2}\right) = F\frac{\varepsilon^2}{4} = 1 \implies$

$\varepsilon = \frac{2}{\sqrt{F}}$, hence we have: $\Delta\delta_{res} = 2\varepsilon = \frac{4}{\sqrt{F}}$

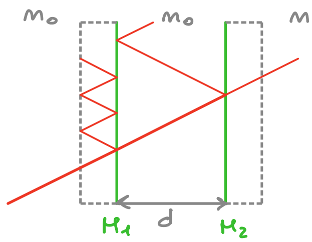
Remember that $\delta = \frac{4\pi}{\lambda_0} n_1 d \cos(\vartheta_i)$, so the phase shift depends on: the wavelength, the refractive index, the thickness and the incident angle.

To build a real F.P. interferometer we usually want to tune the distance d. We can put two mirrors M_1 and M_2 and let the intermediate medium to be air as well.

We have some advantages in using mirrors:

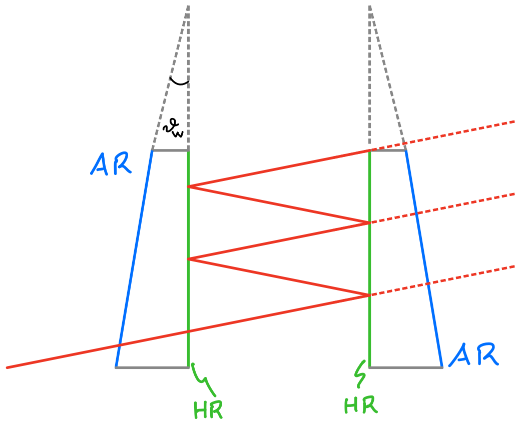
- R_1 and R_2 can be very high using multi-layer dielectric mirrors, so F can be high
- The spacing d between the mirrors can be tuned

Since the mirrors have a thickness, we actually have four interfaces due to substrates.



These parasitic multiple reflections lead to an extra modulation of the Fabry-Perot transmission function.

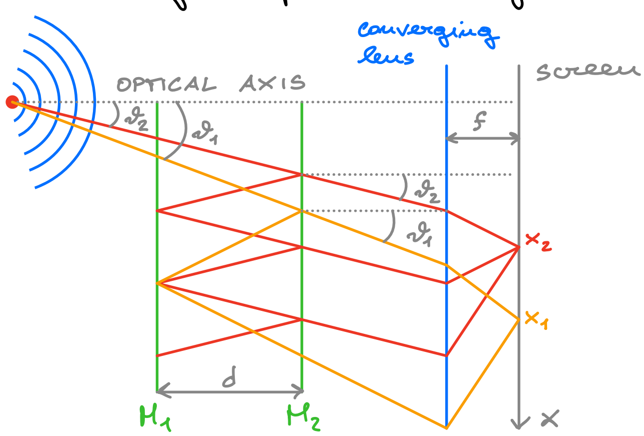
To prevent this issue we use wedged mirrors made with high-reflection coatings and anti-reflection coatings as shown in the Figure aside.



ϑ_w is called wedge angle.

At every bounce the reflection angle is changed. In this way we prevent back reflections and the small reflections will have different angles. We thus have only one main interference pattern.

Let us now study the output of a F.P. interferometer, considering a pointlike light source.



We obtain that

$\vartheta_2 < \vartheta_1 \Rightarrow x_2 < x_1$

For a monochromatic source, in the δ expression the only parameter varying is ϑ.

Hence, there is a 1 to 1 corrispondence between the points of the screen and the phase shift.

We obtain, on the screen, concentric circles (like in the Michelson one). The width of each ring is given by $\Delta\delta_{res}$, so higher values of F bring to thinner rings.

If we have two different wavelengths, the rings due to each λ have a shift. This is the reason why, the lower

$$\Delta\delta_{res} = \frac{4}{\sqrt{F}},$$ the higher the resolution for two different λ.

$$\Delta\delta = \frac{4\tilde{n}}{\lambda_1\lambda_2} d\cos(\vartheta)\cdot\Delta\lambda,$$ where $\Delta\lambda = |\lambda_2 - \lambda_1|$. Hence,

$$\frac{4\tilde{n}}{\lambda^2} d\cos(\vartheta)\,\Delta\lambda_{res} = \frac{4}{\sqrt{F}} \Rightarrow \Delta\lambda_{res} = \frac{\lambda_1\lambda_2}{\tilde{n}d\sqrt{F}} \simeq \frac{\lambda^2}{\tilde{n}d\sqrt{F}}$$

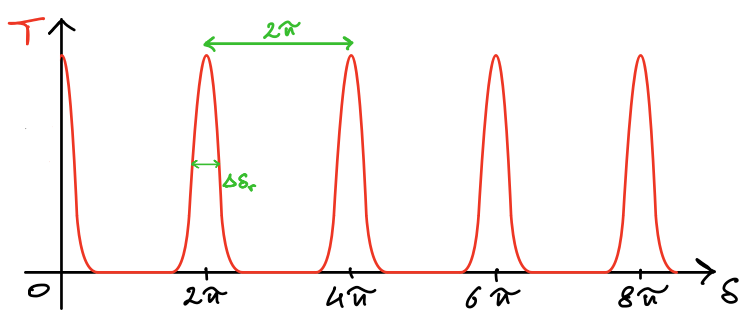
$$\Delta\delta_{FSR} = 2\tilde{n} = \frac{4\tilde{n}}{\lambda^2} d\,\Delta\lambda_{FSR} \Rightarrow \Delta\lambda_{FSR} = \frac{\lambda^2}{2d}$$

$$\frac{\Delta\lambda_{FSR}}{\Delta\lambda_{RES}} = \frac{\tilde{n}\sqrt{F}}{2} = \mathcal{F}$$ is called finesse of the F.P. interferometer

We can also rewrite $\Delta\lambda_{RES} = \dfrac{\Delta\lambda_{FSR}}{\mathcal{F}}$

Let us study the dependence of T_{FP} on the frequency ν of the wave.

$$\delta = \frac{4\tilde{n}}{\lambda} d. \quad \lambda = \frac{c}{\nu} \Rightarrow \delta = \frac{4\tilde{n}d}{c}\nu$$



$$\Delta\delta = \frac{4\tilde{n}d}{c}\Delta\nu \Rightarrow$$

$$\Delta\delta_{res} = \frac{4}{\sqrt{F}} = \frac{4\tilde{n}d}{c}\Delta\nu_{res}$$

$$\Rightarrow \Delta\nu_{res} = \frac{c}{\tilde{n}d\sqrt{F}}$$

$$\Delta\delta_{FSR} = 2\tilde{n} = \frac{4\tilde{n}d}{c}\Delta\nu_{FSR} \Rightarrow \Delta\nu_{FSR} = \frac{c}{2d}$$

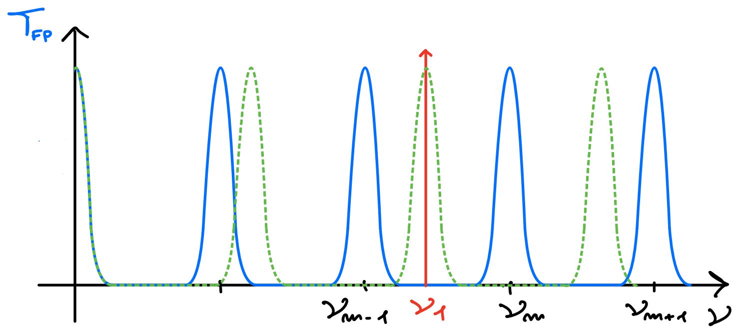
This last result is the free spectral range in the frequency domain

$$\frac{\Delta\nu_{res}}{\Delta\nu_{FSR}} = \frac{c}{\tilde{n}d\sqrt{F}}\cdot\frac{2d}{c} = \frac{2}{\tilde{n}}\cdot\frac{1}{\sqrt{F}} = \frac{1}{\mathcal{F}} \Rightarrow \Delta\nu_{res} = \frac{\Delta\nu_{FSR}}{\mathcal{F}}$$

(remember that $\mathcal{F}$ is the finesse of the F.P. interferometer)

Let us see how to exploit the F.P. interferometer for optical measurements.

Consider a single-frequency laser source with frequency ν_1 and use it as a source for a F.P. interferometer.



$$\nu_m = m\,\Delta\nu_{FSR} = m\,\frac{c}{2d}$$

If ν_1 is out of the peaks (as in the blue plot above) we will see no light at the output.

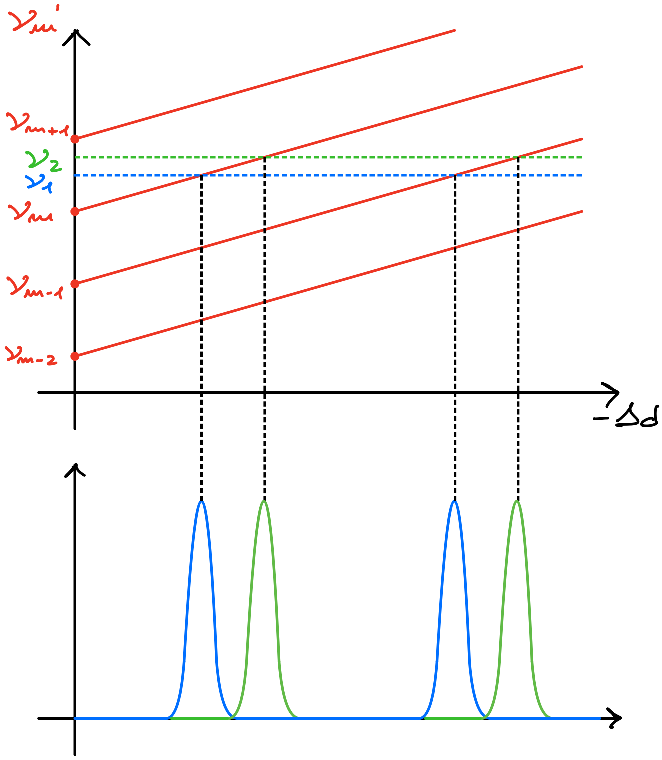
By changing the distance d between the mirrors we can tune the peaks position. The amount of change for the m-th peak is: $\nu_m = m\,\frac{c}{2d} \Rightarrow \Delta\nu_m = \frac{\partial \nu_m}{\partial d}\Delta d = -m\,\frac{c}{2d^2}\Delta d = -\nu_m\,\frac{\Delta d}{d}$

So, the new central peak is at $\nu_m' = \nu_m + \Delta\nu_m = \nu_m\left(1 - \frac{\Delta d}{d}\right)$.

If we have two different frequencies we may see the light from only one source. So, if we have a well known calibration wavelength $\overline{\nu_1} = \nu_m$ we can find m:

$\overline{\nu_1} = \nu_m = m\,\frac{c}{2d} \Rightarrow m = \overline{\nu_1}\cdot\frac{2d}{c}$. Calling $d' = d - \Delta d$, we can find out $\nu_2 = m\cdot\frac{c}{2d'}$. Since d and d' are known, we can find the value of $\nu_2 = \nu_m'$: $\Delta\nu_m = \nu_m' - \nu_m = \nu_2 - \overline{\nu_1} = -\nu_m\,\frac{\Delta d}{d}$

$$\Rightarrow \nu_2 = \overline{\nu_1} - \nu_m\cdot\frac{\Delta d}{d} \Rightarrow \nu_2 = \overline{\nu_1}\left(1 - \frac{\Delta d}{d}\right).$$



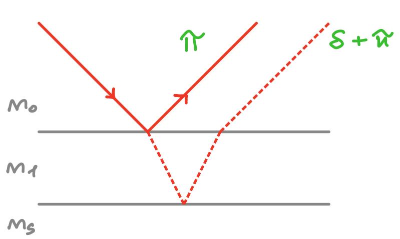
The position of the frequency of the peaks as a function of Δd is shown aside.
Below, we see the intensity on the output screen

If ν_1 and ν_2 are too close they can both be transmitted by the same peak. This gives the resolution of the interferometer: $\nu_2 - \nu_1 \leq \Delta\nu_{RES.}$

Another limitation of the F.P. interferometer is the aliasing: if $\nu_2 - \nu_1 = \Delta\nu_{FSR}$, when we have the m-th peak in correspondance to ν_2, we will have the $(m-1)$-th peak on ν_1. Hence, we can not distinguish the two frequencies.

Multi-layer dielectric coatings

Let us consider the following structure, where $n_0 < n_1 < n_s$



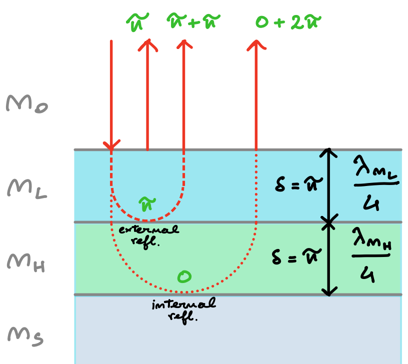
Remember that an external reflection gives a phase shift of π.
Hence, in order to have distructive interference we need:

$$\delta = \pi \implies \delta = \frac{4\pi}{\lambda_{n_1}} d = \pi \implies d = \frac{\lambda_1}{4}$$

For this reason this structure is called quarter-wavelength layer. We find out that

$$R = 0 \quad \text{if} \quad n_1 = \sqrt{n_0 n_S} \implies n_1 = \sqrt{n_S} \quad (\text{if } n_0 \text{ is unitary})$$

In this way we can build single antireflection layers. The issue is that a material with $n_1 = \sqrt{n_S}$ may not exist. What one can do is to use a double AR layer, as sketched



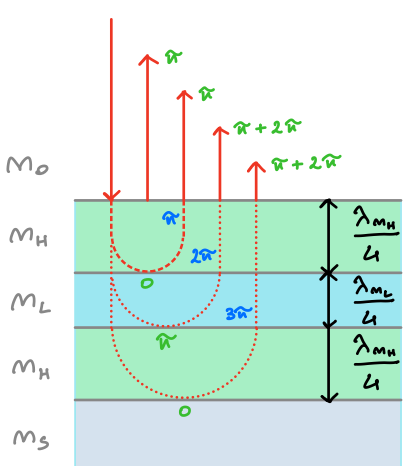
below, with $n_L < n_H$ and $n_H > n_S$.

Let us study the layer with normal incidence.

To build an A.R. layer we have to chose n_L and n_H in order to have

$$R(n_H, n_L, n_S) = 0$$

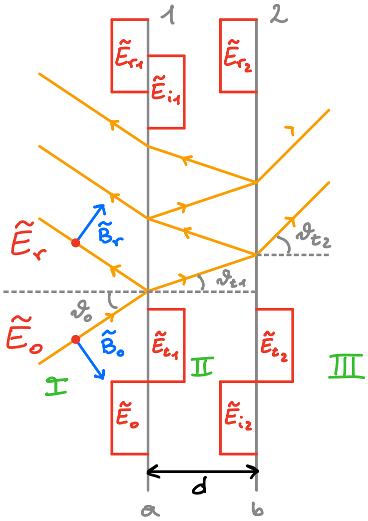
At the same way we can build high reflectivity coatings.



It is sufficient to invert n_H and n_L w.r.t the previous case. In the example aside we use a three-layers structure. All the reflections add up constructively, so we obtain a high reflectivity. This structure is called Bragg reflector.

Let us study the matrix treatment of multi-layer coatings. It is a way to mathematically simulate the reflectivity properties of a multi-layer structure.

Assume to have three layers n_0, n_1 and n_2 and an incident wave with TE polarization



We call $\tilde{E}_{r1}$ the sum of all the electric fields reflected by interface 1 and $\tilde{E}_{t1}$ the sum of all the electric fields transmitted at interface 1. The same convention holds for interface 2.

$\tilde{E}_0$ is the incident field at the first interface.

Let us now apply the preservation of the fields $\tilde{E}$ and $\tilde{B}$ at the two interfaces.

$$\begin{aligned} a: &\ \begin{cases} \tilde{E}_0 + \tilde{E}_{r1} = \tilde{E}_{t1} + \tilde{E}_{i1} \equiv \tilde{E}_a \\ \tilde{E}_{i1} + \tilde{E}_{r2} = \tilde{E}_{t2} \equiv \tilde{E}_b \end{cases} \\ b: & \end{aligned}$$

$$\begin{aligned} a: &\ \begin{cases} \tilde{B}_0 \cos(\vartheta_0) - \tilde{B}_{r1} \cos(\vartheta_0) = \tilde{B}_{t1} \cos(\vartheta_{t1}) - \tilde{B}_{i1} \cos(\vartheta_{t1}) \equiv \tilde{B}_a \\ \tilde{B}_{i2} \cos(\vartheta_{t1}) - \tilde{B}_{r2} \cos(\vartheta_{t1}) = \tilde{B}_{t2} \cos(\vartheta_{t2}) \equiv \tilde{B}_b \end{cases} \\ b: & \end{aligned}$$

Remembering that $\tilde{B} = \dfrac{n}{c_0} \tilde{E}$ for each medium and calling

$$\gamma_0 = \frac{n_0}{c_0} \cos(\vartheta_0); \qquad \gamma_1 = \frac{n_1}{c_0} \cos(\vartheta_{t1}); \qquad \gamma_2 = \frac{n_2}{c_0} \cos(\vartheta_{t2})$$

we can rewrite the equations on the magnetic field as:

$$\begin{cases} \tilde{E}_0 \gamma_0 - \tilde{E}_{r1} \gamma_0 = \tilde{E}_{t1} \gamma_1 - \tilde{E}_{i1} \gamma_1 = \tilde{B}_a \\ \tilde{E}_{i2} \gamma_1 - \tilde{E}_{r2} \gamma_1 = \tilde{E}_{t2} \gamma_2 = \tilde{B}_b \end{cases} \implies$$

$$\begin{cases} \gamma_0 (\tilde{E}_0 - \tilde{E}_{r1}) = \gamma_1 (\tilde{E}_{t1} - \tilde{E}_{i1}) = \tilde{B}_a \\ \gamma_1 (\tilde{E}_{i2} - \tilde{E}_{r2}) = \gamma_2 \tilde{E}_{t2} = \tilde{B}_b \end{cases}$$

Note that we have seven variables and four equations. Hence, we should find three more relations.

We could write $\tilde{E}_{i2} = \tilde{E}_{t1} e^{-i\delta}$ and $\tilde{E}_{i1} = \tilde{E}_{r2} e^{-i\delta}$, where

$\delta = \frac{2\pi}{\lambda_0} n_1 d \cos(\vartheta)$ (since it is for a distance d and not for a distance 2d as usual).

Now, knowing $\tilde{E}_0$, we can find all the fields as a function of the incident electric field. Putting together the three sets of equations we obtain

$$\begin{bmatrix} \tilde{E}_a \\ \tilde{B}_a \end{bmatrix} = \underbrace{\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}}_{M} \begin{bmatrix} \tilde{E}_b \\ \tilde{B}_b \end{bmatrix} = \begin{bmatrix} \cos\delta & i\frac{\sin\delta}{\gamma_1} \\ i\gamma_1 \sin\delta & \cos\delta \end{bmatrix} \begin{bmatrix} \tilde{E}_b \\ \tilde{B}_b \end{bmatrix}$$

Let us now consider a general multilayer stack with N layers of length $d_1, \dots, d_N$ and refractive index $n_1, \dots, n_N$. The j-th layer is described by a matrix M_j. Hence, we can write

$$\begin{bmatrix} \tilde{E}_{aj} \\ \tilde{B}_{aj} \end{bmatrix} = M_j \begin{bmatrix} \tilde{E}_{bj} \\ \tilde{B}_{bj} \end{bmatrix}, \quad \text{with} \quad \begin{bmatrix} \tilde{E}_{bj} \\ \tilde{B}_{bj} \end{bmatrix} = M_j \begin{bmatrix} E_{a,j+1} \\ B_{a,j+1} \end{bmatrix}$$

Thus, if we know the transfer matrix of each layer we can write the overall transfer matrix as the product of all the transfer matrices:

$$\begin{bmatrix} \tilde{E}_{a1} \\ \tilde{B}_{a1} \end{bmatrix} = M_{TOT} \begin{bmatrix} \tilde{E}_{bN} \\ \tilde{B}_{bN} \end{bmatrix} \Rightarrow \begin{bmatrix} \tilde{E}_{a1} \\ \tilde{B}_{a1} \end{bmatrix} = \prod_{j=1}^{N} M_j \begin{bmatrix} \tilde{E}_{bN} \\ \tilde{B}_{bN} \end{bmatrix};$$

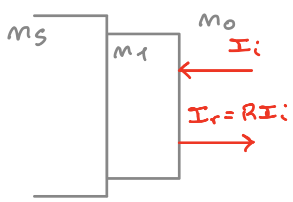
This method is very useful to implement the mathematical description on a numerical simulator.

The reflection coefficient of a given transfer matrix M_{TOT} is

$$r \equiv \frac{\tilde{E}_r}{\tilde{E}_0} = \frac{\gamma_0 m_{11} + \gamma_0 \gamma_2 m_{12} - m_{21} - \gamma_2 m_{22}}{\gamma_0 m_{11} + \gamma_0 \gamma_2 m_{12} + m_{21} + \gamma_2 m_{22}}$$

$R = |r|^2 = r \cdot r^*$ and $T = 1 - R$.

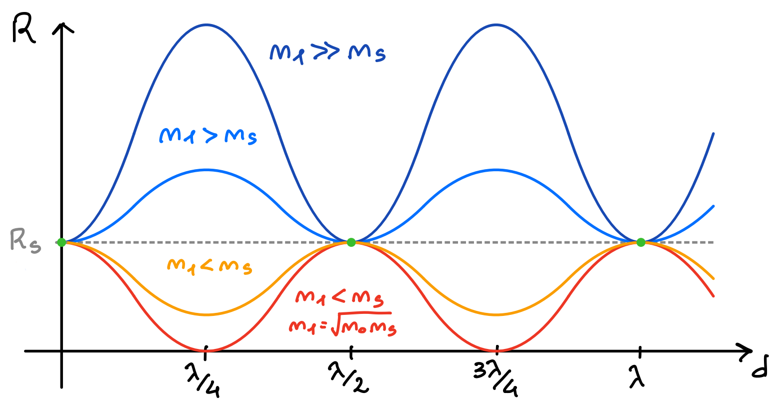
Example 1: single layer with normal incidence.



$$\gamma_0 = \frac{n_0}{c_0}; \quad \gamma_1 = \frac{n_1}{c_0}; \quad \gamma_2 = \frac{n_S}{c_0}; \quad \delta = \frac{2\pi}{\lambda_0} n_1 d$$

$$M = \begin{bmatrix} \cos\delta & i\frac{c_0}{n_1}\sin\delta \\ i\frac{c_0}{n_1}\sin\delta & \cos\delta \end{bmatrix}$$

$$R = rr^* = \frac{n_1(n_0 - n_S)^2\cos^2\delta + (n_0 n_S - n_1)^2\sin^2\delta}{n_1(n_0 - n_S)^2\cos^2\delta + (n_0 n_S + n_1)^2\sin^2\delta}$$



$$R_S = \left(\frac{n_0 - n_S}{n_0 + n_S}\right)^2$$

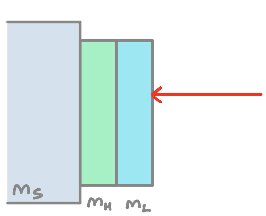
We can reach $R = 0$ if

$d = \lambda/4 \Rightarrow \delta = \frac{\pi}{2}$. So, the transfer matrix is

$$M_1 = \begin{bmatrix} 0 & j\frac{c_0}{n_1} \\ j\frac{n_1}{c_0} & 0 \end{bmatrix}, \quad \text{since } \sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{\pi}{2} = 0$$

To have $R = 0$ we must also have $n_1 = \sqrt{n_0 n_S}$

Example 2: double layer AR coating in normal incidence



We saw previously that we have to put

$$d_H = \frac{\lambda_0}{4 n_H} \quad \text{and} \quad d_L = \frac{\lambda_0}{4 n_L}$$

In normal incidence $\cos\delta = 0$ and $\sin\delta = 1$, hence

$$M_L = \begin{bmatrix} 0 & i\dfrac{c_0}{m_L} \\ i\dfrac{m_L}{c_0} & 0 \end{bmatrix} \quad \text{and} \quad M_H = \begin{bmatrix} 0 & i\dfrac{c_0}{m_H} \\ i\dfrac{m_H}{c_0} & 0 \end{bmatrix}$$

$$\text{So,} \quad M_{DOUBLE} = M_L \cdot M_H = \begin{bmatrix} -\dfrac{m_H}{m_L} & 0 \\ 0 & -\dfrac{m_L}{m_H} \end{bmatrix} \Longrightarrow$$

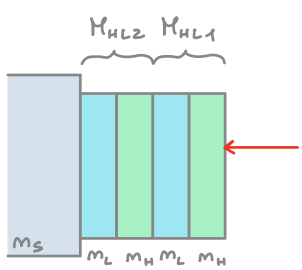
Let us impose the reflectivity equal to zero to find m_H, m_L

$$r = \frac{\dfrac{m_0}{c_0}\left(-\dfrac{m_H}{m_L}\right) - \dfrac{m_S}{c_0}\left(-\dfrac{m_L}{m_H}\right)}{\dfrac{m_0}{c_0}\left(-\dfrac{m_H}{m_L}\right) + \dfrac{m_S}{c_0}\left(-\dfrac{m_L}{m_H}\right)} = 0 \Longrightarrow \frac{m_0}{m_S} = \left(\frac{m_L}{m_H}\right)^2$$

Hence, $R = 0 \Longleftrightarrow \dfrac{m_L}{m_H} = \sqrt{\dfrac{m_0}{m_S}}$. If $m_0 = 1$ we obtain $\dfrac{m_H}{m_L} = \sqrt{m_S}$.

This is, in practice, an easier condition to fulfill w.r.t. the single layer case

Example 3: multilayer HR coatings



$$M_{HL} = \begin{bmatrix} 0 & i\dfrac{c_0}{m_H} \\ i\dfrac{m_H}{c_0} & 0 \end{bmatrix} \begin{bmatrix} 0 & i\dfrac{c_0}{m_L} \\ i\dfrac{m_L}{c_0} & 0 \end{bmatrix} = \begin{bmatrix} -\dfrac{m_L}{m_H} & 0 \\ 0 & -\dfrac{m_H}{m_L} \end{bmatrix}$$

Assuming to have N double layers, we find the total transfer matrix as

$$M_{TOT} = M_{HL}^N = \begin{bmatrix} -\dfrac{m_L}{m_H} & 0 \\ 0 & -\dfrac{m_H}{m_L} \end{bmatrix}^N = \begin{bmatrix} \left(-\dfrac{m_L}{m_H}\right)^N & 0 \\ 0 & \left(-\dfrac{m_H}{m_L}\right)^N \end{bmatrix}$$

$$r = \frac{m_0\left(-\dfrac{m_L}{m_H}\right)^N - m_S\left(-\dfrac{m_H}{m_L}\right)^N}{m_0\left(-\dfrac{m_L}{m_H}\right)^N + m_S\left(-\dfrac{m_H}{m_L}\right)^N} \Longrightarrow R = |r|^2 = \left[\frac{\left(\dfrac{m_0}{m_S}\right)\left(\dfrac{m_L}{m_H}\right)^{2N} - 1}{\left(\dfrac{m_0}{m_S}\right)\left(\dfrac{m_L}{m_H}\right)^{2N} + 1}\right]^2$$

Since $\frac{m_L}{m_H} < 1$, $\left(\frac{m_L}{m_H}\right)^{2N} \longrightarrow 0$ for N high. So for N high $R \longrightarrow 1$

3. LASERS WORKING PRINCIPLE

Properties of Laser light

The main properties that characterize an ideal laser light are:

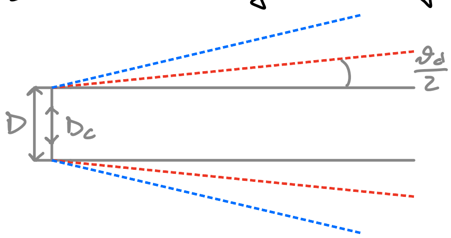
1) Monochromaticity: high temporal coherence.

   Since $S(\nu) = \mathcal{F}(\Gamma(\tau))$ we have that $\Delta\nu_{FWHM}$ is narrow

   Nowadays we can reach frequency purities up to $\dfrac{\Delta\nu_{FWHM}}{\nu_0} = 10^{-18}$.

2) Ultrashort pulses. The duration of a pulse is proportional to the reciprocal of the FWHM: $\Delta\tau_P = \dfrac{1}{\Delta\nu_{FWHM}}$

3) Low divergence: high spatial coherence. In general, the



minimum divergence angle is $\vartheta_d = \dfrac{\lambda}{D}$, where D is the diameter.

So, the larger is the diameter of the beam, the smaller is the divergence angle. The former relation holds only if there is a perfect spatial coherence. In this case the beam is called diffraction limited.

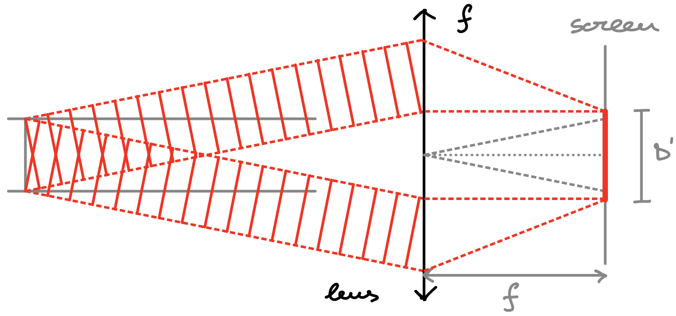
Laser are always diffraction limited sources.

If a source does not have a perfect spatial coherence, the diameter of the area of coherence D_c is smaller than the diameter of the emitter: $D_c < D$. In this case we can write

$$\vartheta_d = \frac{\lambda}{D_c}$$

4) High focusability, given by low divergence and high spatial coherence.

Let us consider a laser source with a divergence angle and a lens with focal length f. In the figure below the two plane waves with maximum ϑ_d are shown

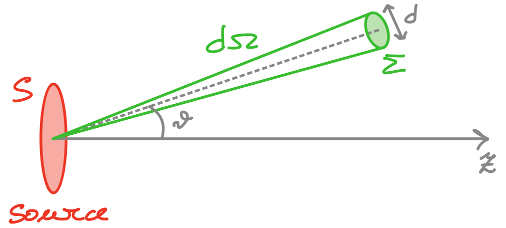


$$\begin{cases} D' = \vartheta_d f \\ \vartheta_d = \dfrac{\lambda}{D} \end{cases} \implies D' = f \frac{\lambda}{D}$$

The maximum value of the beam diameter is $D = D_L$, where D_L is the diameter of the lens. If this happens we can write

$D' = \lambda \dfrac{f}{D_L} = \lambda \cdot N.A.$, where $N.A. = \dfrac{f}{D_L}$ is the numerical aperture

5) Very high brillance. The brillance represents the brightness of a source.



Consider the structure in figure, where S is the surface of the emitter

The infinitesimal power dP emitted on an angle ϑ is given by: $dP(\vartheta) = B \cos(\vartheta) S \cdot d\Omega$,

where B is a proportionality constant called brillance and Ω is the solid angle. Note that $B \propto \dfrac{1}{S} \dfrac{dP}{d\Omega}$

For a laser beam, $\vartheta \simeq 0 \implies \cos(\vartheta) \simeq 1$ and B is a constant w.r.t ϑ (the source is Lambertian)

The total power emitted by a laser beam is thus given by

$$P = B S \Omega \implies B = \frac{P}{S\Omega} = \frac{P z^2}{S \Sigma} = \frac{P}{S} \cdot \frac{4}{\pi} \frac{d^2}{z^2} .$$

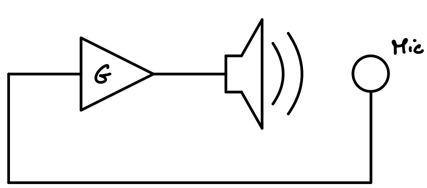
But $\vartheta_d = \dfrac{d}{z} \implies \Omega = \dfrac{\pi}{2} \vartheta_d^2$ and $S = \pi \left(\dfrac{D}{2}\right)^2 = \dfrac{\pi D^2}{4}$.

Since $\vartheta_d = \dfrac{\lambda}{D}$, we can eventually write $B = \dfrac{P}{\pi \dfrac{D^2}{4} \cdot \dfrac{\pi}{4} \dfrac{\lambda^2}{D^2}}$

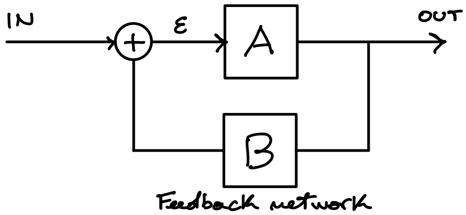
$$\implies B = \frac{P}{\left(\frac{\pi}{4} \cdot \lambda\right)^2}$$

Principle of operation

The acronym LASER stands for Light Amplification by Stimulated Emission of Radiation.
Hence, a laser is an amplificator of light. The working principle is the following: given a power supply we obtain an output without any light input. Hence, a laser is not actually an amplifier, but is an oscillator. The principle is similar to the Larsen effect in audio, hence it is a positive feedback that tends to saturate the system.



Not all the frequencies are amplified, but only the integer multiples of $1/\tau$: $f = m \cdot \frac{1}{\tau}$, i.e. all the frequencies that interfer constructively with themselves after a round inside the system.
Let us consider the block diagram of a positive feedback



$\varepsilon = IN + B \cdot OUT$; $\quad OUT = \varepsilon A$

In a positive feedback, ideally $OUT \neq 0$ even if $IN = 0$

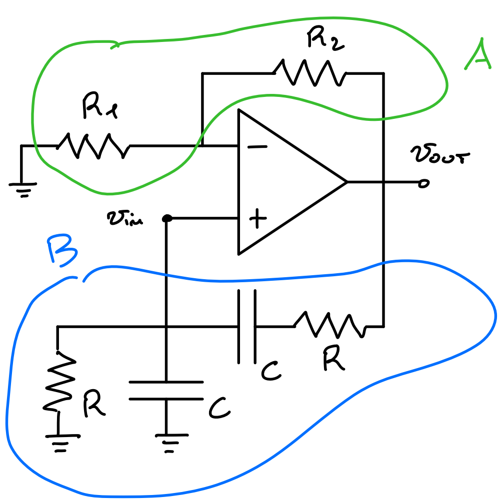
$$OUT = A(IN + B \cdot OUT) \implies OUT = \frac{A}{1-AB} \cdot IN, \text{ where } AB = G_{loop}.$$

Hence the relation is the same of the negative feedback one, but in a positive feedback $A/(1-AB) \to \infty$, thus we must have

$$\begin{cases} |G_{loop}| = 1 \\ \measuredangle[G_{loop}] = 2m\pi \end{cases} \quad \text{(Barkhausen criteria)}$$

A laser is an optical amplifier with a positive feedback system that satisfies the Barkhausen conditions for at least one optical frequency ν_L.

Let us consider, as an analogy, the Wien-Bridge oscillator



$$A = \frac{v_{out}}{v_{in}} = 1 + \frac{R_2}{R_1}$$

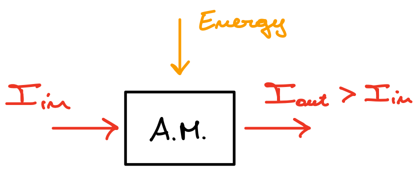
$$B = \frac{v_{in}}{v_{out}} = \frac{s/RC}{s^2 + \frac{3}{RC}s + \frac{1}{(RC)^2}}$$

$s = \alpha + j\omega$. At regime $\alpha = 0$, so

$$G_{loop} = AB = \frac{\left(1+\frac{R_2}{R_1}\right) \cdot \frac{j\omega}{RC}}{-\omega^2 + j\frac{3}{RC}\omega + \frac{1}{(RC)^2}}$$

$$\begin{cases} |G_{loop}| = 1 \implies |G_{loop}| = \frac{1}{3}\left(1+\frac{R_2}{R_1}\right) \implies R_2 = 2R_1 \\ \measuredangle[G_{loop}] = 2m\pi \underset{m=0}{\implies} \omega = \frac{1}{RC} = \omega_0 \quad \text{(resonance frequency)} \end{cases}$$

At the same way, we start from an optical amplifier to build an optical oscillator. An optical amplifier can be schematized as follows, where A.M. stands for amplifier



material. The input energy is given by a pumping system.

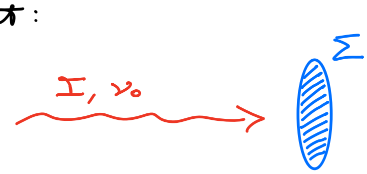
To build an oscillator we need a positive feedback, that, in an optical system, is given by a system of mirrors.

The gain element is given by the active material.

Interaction between radiation and matter

Studying the interaction between radiation and matter, we have to take into account that:

1) Radiation is quantized

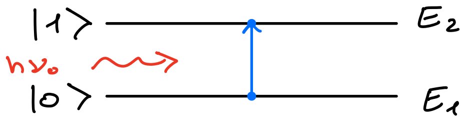


$$P = I \Sigma = \frac{\Delta E_\Sigma}{\Delta t},$$ where $\Delta E_\Sigma = m h \nu_0$. m is the number of photons that crosses the surface Σ in a time Δt

$$I = \frac{P_\Sigma}{\Sigma} = \frac{\Delta E_\Sigma}{\Delta t \cdot \Sigma} = \frac{m h \nu_0}{\Delta t \cdot \Sigma} = \Phi \cdot h \nu_0,$$

where Φ is the photon flux.

2) Matter is quantized. The amount of energy that can be absorbed is quantized. Hence, the atoms can interact with radiation in different ways.



If the interaction is resonant, i.e. $h\nu_0 = E_2 - E_1$, the photon is absorbed and a molecule passes to the excited state. This is an absorption process

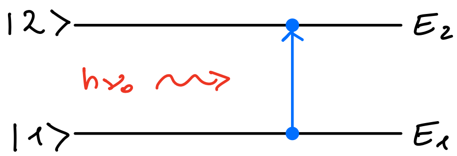
Let's call N_1 and N_2 the population of two electronic levels.

In thermodynamic equilibrium we know that $N_K \propto e^{\frac{E_K}{KT}}$.

Thus: $$\frac{N_2}{N_1} \propto \exp\left[\frac{E_2 - E_1}{KT}\right] = e^{\frac{\Delta E_{21}}{KT}} = e^{\frac{h\nu_0}{KT}}$$

If ν_0 is an optical frequency, $\Delta E_{21} \gg KT$ since $h\nu_0 > 2\,eV$ while $KT \simeq 26\,meV$. Hence, we can write

$N_2 = N_1\, e^{-\frac{\Delta E_{21}}{KT}} \implies N_2 \ll N_1$. So, for optical transitions the lower level is always much more occupied than the higher at thermodynamic equilibrium.



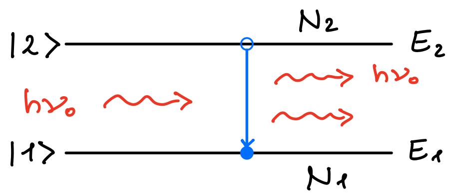
The velocity at which the absorption process occurs can be found from the following equation that will be proved later during the course:

$$\frac{dN_1}{dt} = -W_{12} N_1 .$$

W_{12} is called transition rate from $|1\rangle$ to $|2\rangle$

Hence, $|W_{12}| = \frac{1}{dt}\left|\frac{dN_1}{dt}\right|$, so it is the fractional change of population N_1 per unit time dt

The other process that can take place is the stimulated emission process.



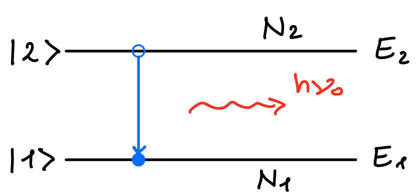
Let us assume that $N_1 \gg N_2$, but $N_2 \neq 0$. If a photon $h\nu_0$ resonant with the transition $E_2 - E_1$ hits an atom in $|2\rangle$, there is a finite probability that another photon is emitted. The emitted photon has the same direction, frequency and phase of the stimulating one. This last property means that the emitted photon contributes coherently to the increase of amplitude of the electromagnetic field: it is an amplifying process. In this case we can write

$$\frac{dN_2}{dt} = -W_{21} N_2, \qquad W_{21} \text{ is the stimulated emission rate.}$$

Remark: For a given material interacting with an E.M. field, $W_{21} = W_{12}$. We will prove that they can be expressed by $W_{12} = W_{21} = \sigma_{12} \cdot \Phi_{ph}$. σ_{12} is the absorption cross-section.

Hence, $\frac{dN_1}{dt} = -W_{12} N_1 = -\sigma_{12} \Phi_{ph} N_1$.

The third kind of process is the spontaneous emission.

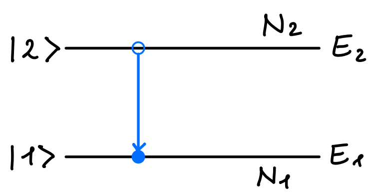


A spontaneous transition from $|2\rangle$ to $|1\rangle$ generates a photon with energy $h\nu_0 = E_2 - E_1$.

If an E.M. field is present, the emitted photon is completely uncorrelated to the photons of the external field; hence, these photons do not contribute to the amplification process.

$$\left.\frac{dN_2}{dt}\right|_{sp} = -AN_2 = -\frac{N_2}{\tau_{sp}}$$

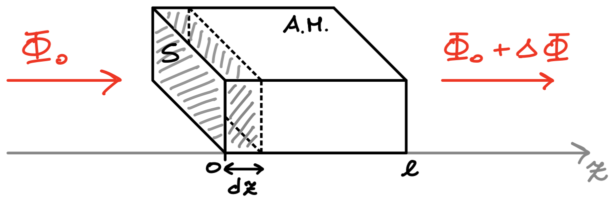
Eventually, the fourth process is the non radiative decay



In this case, there is a decay $|2\rangle \rightarrow |1\rangle$ without emitting any photon.

In this case $\Delta E_{21} = E_2 - E_1$ is released increasing the molecular kinetic energy (if the material is in gas phase) or increasing the vibrational energy of the atomic lattice (if the matirial is in solid state).

Let us now see how to achieve optical amplification considering an amplifying material as shown in figure.



$d\Omega = S \cdot dz$

The total number of photons that crosses S per unit time is given by $\Phi_0 \cdot S$. The number of photons added or removed to the E.M. field is $d\Phi \cdot S$.

Considering a stimulated emission process we can write

$$\begin{cases} d\Phi \cdot S = \left(-\dfrac{dN_2}{dt}\right)_{SE} d\Omega - \left(-\dfrac{dN_1}{dt}\right)_{ABS} d\Omega \\ d\Omega = S\,dz \end{cases} \Longrightarrow$$

$$\begin{cases} S \cdot d\Phi = \left(-\dfrac{dN_2}{dt}\right)_{SE} S\,dz + \left(\dfrac{dN_1}{dz}\right)_{ABS} \cdot S\,dz \\ \left(\dfrac{dN_2}{dt}\right)_{SE} = -\sigma_{21}\,\Phi\,N_2\,; \quad \left(\dfrac{dN_1}{dt}\right)_{ABS} = -\sigma_{21}\,\Phi\,N_1\,; \quad \sigma_{21} = \sigma_{12} = \sigma \end{cases}$$

We obtain the following differential equation

$\dfrac{d\Phi}{dz} = (N_2 - N_1)\,\sigma\,F$. Defyning $\Delta N = N_2 - N_1$ and integrating

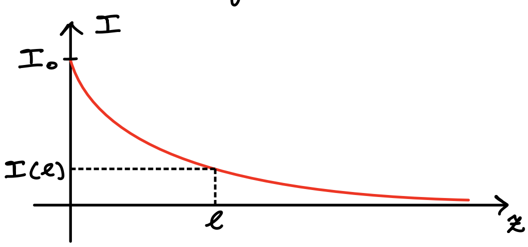
$$\int_{F_0}^{F} \frac{dF}{F} = \int_0^{\ell} \Delta N\,\sigma\,\Phi\,dz \Rightarrow \ln\left(\frac{F}{F_0}\right) = \Delta N\,\sigma\,\ell \Rightarrow$$

$\Phi(\ell) = \Phi_0\,e^{\Delta N \sigma \ell}$ (photon flux at the output)

Since $I \propto \Phi$, $I(\ell) = I_0\,e^{\Delta N \sigma \ell}$

At thermodynamic equilibrium $N_1 > N_2 \Rightarrow \Delta N \sigma \ell < 0$;

we can define $\Delta N \sigma = -\alpha$, $\alpha > 0$. In this case we have



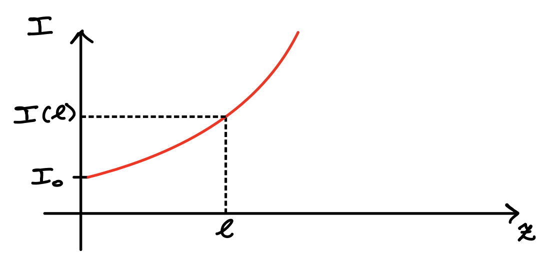
$I(\ell) = I_0\,e^{-\alpha\ell}$, that is called

Lambert - Beer law

If a way to have $\Delta N > 0$ is provided by an opportune pumping system, we have $\Delta N \cdot \sigma > 0$.

We define $g = \Delta N \cdot \sigma$ as the gain per unit length of the active material

The intensity will be given by



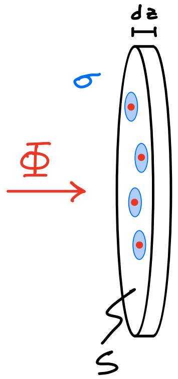
$I(\ell) = I_0\,e^{g\ell}$

Physical meaning of the cross-sectional area σ.

Let us assume to be at 0 Kelvin. The only process that can take place is the absorption, with equation

$$\frac{dN_2}{dt} = -\sigma \Phi N_1 . \quad N_2 = 0 \Rightarrow \Delta N = -N_1, \text{ so } \quad \Phi(\ell) = \Phi_0 e^{-N_1 \sigma \ell}$$

Let us consider a very thin material of surface S.



The whole population N_1 is absorbing photons. The number of absorbers is given by

$m_{abs} = N_1 \, d\Omega = N_1 \, S \, dz$

σ is the area surrounding each absorber such that if a photon hits the area σ the photon is absorbed with probability $P = 1$

The overall absorbing area is $m_{abs} \cdot \sigma$

$$\frac{d\Phi}{\Phi} = -\frac{m_{abs}\,\sigma}{S} = -\frac{N_1 S \, dz \, \sigma}{S} \Rightarrow \frac{d\Phi}{\Phi} = -N_1 \sigma \, dz$$

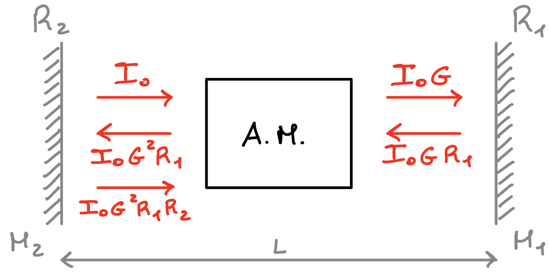
that is the equation we found before.

We know that, according to the Boltzmann statistics, at thermodyinamic equilibrium an active material behaves as an absorber.

We saw that, to obtain an increasing of the intensity we need a population inversion $(N_2 > N_1 \Rightarrow \Delta N > 0)$, achieving

$$I(\ell) = G I_0 = e^{g\ell} I_0$$

A basic configuration of the mirrors to obtain a positive feedback is the mirror cavity, as shown below. Assume that the mirrors have non-unitary reflectivities.



After one loop we obtain an intensity $I(2L) = I_0 G^2 R_1 R_2$

Actually, other losses should be considered, for example the non-zero reflectivity of the material interface or photons scattering inside the material. Hence, calling L_i the total internal losses in the cavity the intensity becomes

$$I(2L) = I_0 G^2 R_1 R_2 (1-L_i)^2$$

Remember that the Barkhausen conditions must be satisfied, in particular $|G_{loop}| = 1$. So:

$$G_{loop} = \frac{I(2L)}{I_0} = G^2 R_1 R_2 (1-L_i)^2 = 1 = e^{2\sigma \Delta N \ell} R_1 R_2 (1-L_i)^2 = 1$$

$$\Rightarrow \ln(G_{loop}) = 2\sigma\Delta N \ell + \ln(R_1) + \ln(R_2) + 2\ln(1-L_i) = 0$$

Since $R_1, R_2 < 1$, $\ln(R_1), \ln(R_2) < 0$. Calling $\ln(R_1) = -\gamma_1$; $\ln(R_2) = -\gamma_2$ and $\ln(1-L_i) = -\gamma_i$ (internal logarithmic losses). Hence we can rewrite:

$$2\sigma\Delta N \ell - 2\left[\frac{\gamma_1+\gamma_2}{2} + \gamma_i\right] = 0$$

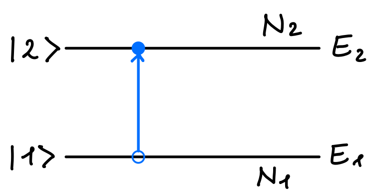
Remember that $R_1 = 1 - T_1$, $T_1 \simeq 0$, hence we can linearize the logarithm as $\ln(1-T_1) \underset{T_1 \to 0}{\sim} -T_1$, thus

$\gamma_1 = -\ln(1-T_1) \simeq T_1$. Calling $\gamma = \left[\dfrac{\gamma_1+\gamma_2}{2} + \gamma_i\right]$, that represents the average losses per single pass of the cavity, we can write the Barkhausen condition as:

$\sigma \Delta N \ell - \gamma = 0 \implies \Delta N_c = \dfrac{\gamma}{\sigma \ell}$. ΔN_c is the critical value of the population inversion, i.e. the threshold for laser action.

Pumping systems

Let us now see how to efficiently put the system out of thermodynamical equilibrium obtaining $N_2 > N_1$.



Let us call pumping rate the rate at which we pump the level $|2\rangle$:

$$R_p = \frac{dN_2}{dt}$$

If we have an optical pumping system, we shine a photon flux Φ with photon energy $h\nu_p = E_2 - E_1 = h\nu_0$

$$R_p = \left.\frac{dN_2}{dt}\right|_{PUMP} = \sigma \Phi N_1 = -\left.\frac{dN_1}{dt}\right|_{ABS}$$

$$\frac{dN_2}{dt} = \left.\frac{dN_2}{dt}\right|_{PUMP} + \left.\frac{dN_2}{dt}\right|_{SE} = R_p - \sigma\Phi N_2 = \sigma\Phi N_1 - \sigma\Phi N_2 = \sigma\Phi(N_1 - N_2)$$

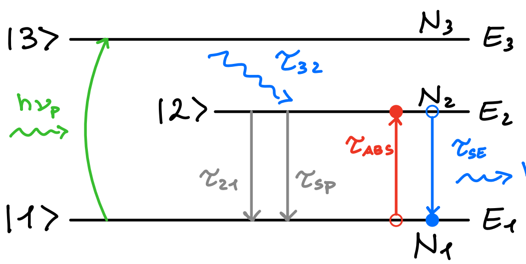
$$\Rightarrow \frac{dN_2}{dt} = \sigma\Phi(N_t - 2N_2), \quad \text{where } N_t = N_1 + N_2$$

The solution of the former differential equation is an exponential function with asymptotic value

$$\lim_{t\to\infty} N_2(t) = N_1 = \frac{N_t}{2}.$$

So, with a two-level system only the transparency condition $\left(\Delta N = N_2 - N_1 = 0 \implies N_1 = N_2 = \frac{N_t}{2}\right)$

Let us now see what happens with a three-level system. From now on we will indicate with $|1\rangle$ the lower laser level and $|2\rangle$ the upper laser level. A third state $|3\rangle$ is called pumping level and is coupled with $|2\rangle$ through a very quick non-radiative decay.



$$\left.\frac{dN_3}{dt}\right|_{NR} = \frac{N_3}{\tau_{32}},$$ where τ_{32} is the non-radiative decay time of the transition $|3\rangle \to |2\rangle$;

We can always assume $\tau_{32} \ll \tau_{sp}, \tau_{21}$ that are, respectively, the decay times of spontaneous emission $|2\rangle \to |1\rangle$ and non radiative decay of $|2\rangle \to |1\rangle$.

Now pump photons are different from laser photons: $h\nu_p \neq h\nu_0$.

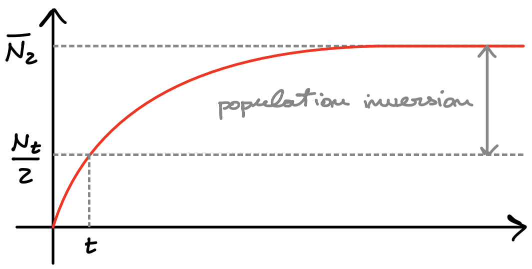
The pump rate is $R_p = \frac{dN_3}{dt} = \left.\frac{dN_2}{dt}\right|_{PUMP}$ (since $\tau_{32} \simeq 0$)

The laser action is given by the stimulated emission $|2\rangle \to |1\rangle$. Neglecting the spontaneous and the non-radiative emissions we can eventually write

$$\frac{dN_2}{dt} = \left.\frac{dN_2}{dt}\right|_{PUMP} + \left.\frac{dN_2}{dt}\right|_{SE} + \left.\frac{dN_2}{dt}\right|_{ABS} = R_p - \sigma N_2 \Phi + \sigma N_1 \Phi$$

$$\Rightarrow \frac{dN_2}{dt} = R_p - \sigma \Phi (N_2 - N_1) = R_p - \sigma \Phi (2N_2 - N_t)$$

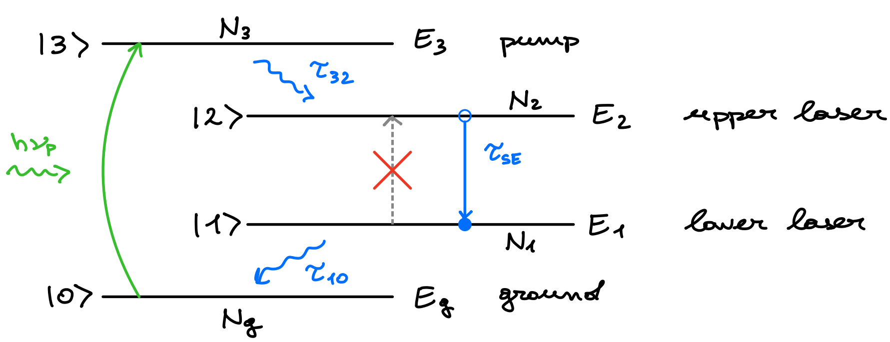
$$\frac{dN_2}{dt} = 0 \Rightarrow R_p - \sigma \Phi (2N_2 - N_t) = 0$$



The asymptotic value $\overline{N_2}$ for $t \to \infty$ is given by

$$\begin{cases} \overline{N_2} = \dfrac{N_t}{2} + \dfrac{R_p}{2\sigma\Phi} \\ \overline{N_1} = \dfrac{N_t}{2} - \dfrac{R_p}{2\sigma\Phi} \end{cases} \Rightarrow \Delta N = \frac{R_p}{\sigma \Phi}$$

Eventually, we can consider four level systems that is even more efficient than the three level system.



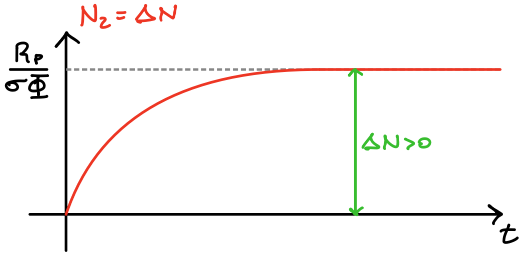
τ_{32} and τ_{10} are non radiative and very small. For this reason $N_3 \simeq 0$ and $N_1 \simeq 0$, hence $\Delta N = N_2$, that means that we have population inversion even with one molecule in N_2.

Furthermore we can not have absorption from N_1 to N_2 since $\left.\frac{dN_1}{dt}\right|_{ABS} = \sigma \Phi N_1 = 0$

The differential equation in this case becomes

$$\frac{dN_2}{dt} = R_p - \sigma \Phi N_2$$

At regime ($t \to \infty$) we obtain $R_P - \sigma \Phi N_2 = 0 \Rightarrow \bar{N}_2 = \dfrac{R_P}{\sigma\Phi} = \Delta N$

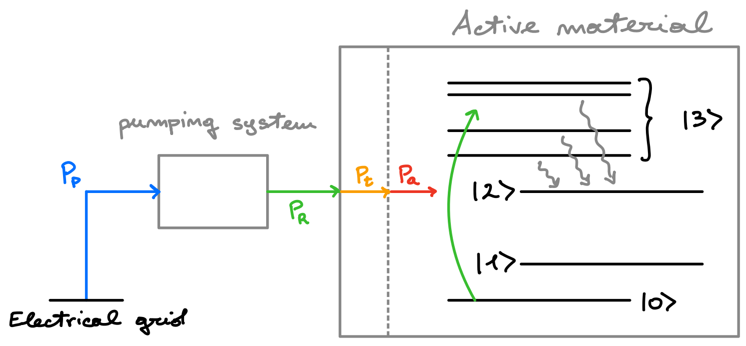


Note that if the photon flux increases $\bar{N}_2$ decreases.

It is straightforward to understand that the four-level laser is the most efficient one.

Let us now study the efficiency of a pumping system. It is defined as the ratio between the power transmitted to the minimum pumping level for the system and the power absorbed by the pump:

$$\eta_P = \frac{P_m}{P_P}$$



P_R is the radiative power emitted by the pumping system. The radiative efficiency of the system is $\eta_R = \dfrac{P_R}{P_P}$

Only a fraction P_T of P_R enters the A.M. The transfer efficiency is $\eta_T = \dfrac{P_T}{P_R}$.

The third efficiency term is given by the power absorbed by the material P_a: $\eta_a = \dfrac{P_a}{P_t}$ (absorption efficiency).

P_m is the minimum pumping power needed if $|3\rangle$ was degenerate in energy with $|2\rangle$. In that case:

$E_3 = E_2 \Rightarrow h\nu_p = E_2 - E_g = h\nu_{amp}$. In a real case we have that $h\nu_p = E_3 - E_g > E_2 - E_g \Rightarrow h\nu_p > h\nu_{amp}$. So, $P_a > P_m$.

Hence, the minimum pumping quantum efficiency is

$\eta_{mq} = \frac{P_m}{P_a}$ (P_a is the power absorbed by the A.M.)

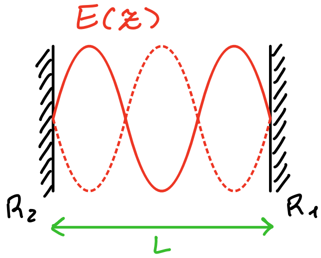
Finally, the overall pumping efficiency is given by

$$\eta_p = \eta_R \eta_t \eta_a \eta_{mq} = \frac{P_R}{P_p} \cdot \frac{P_t}{P_R} \cdot \frac{P_a}{P_t} \cdot \frac{P_m}{P_a} = \frac{P_m}{P_p}$$

Optical resonator

An optical resonator is an optical feedback system designed in order to be an oscillator.

The most simple optical resonator is made by a couple of parallel facing mirrors (in practice it is a Fabry-Perot interferometer). Let us consider metallic mirrors (so we have $\vec{E} = \underline{0}$ on the surfaces).



$$E(z) = E_0 e^{jkz} \Rightarrow E(2L) = \alpha E_0 e^{jk \cdot 2L}$$

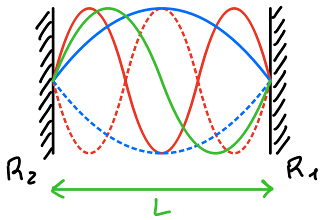
In order to have that the field interfers constructively with itself, we must have $\angle[E(2L)] = k \cdot 2L = 2\pi m$

So, the resonance condition is the Barkhausen condition:

$$G_{loop} = \frac{E(2L)}{E_0} = \alpha e^{jk \cdot 2L} \Rightarrow \angle[G_{loop}] = 2kL = 2\pi m$$

$$k = \frac{2\pi}{\lambda_m} \cdot 2L = 2m\pi \Rightarrow \lambda_m = \frac{2L}{m} \Rightarrow L = m \cdot \frac{\lambda_m}{2}$$

Hence, the wavelengths that are allowed in a cavity of length L are: $\lambda_1 = 2L$; $\lambda_2 = L$, $\lambda_3 = \frac{3}{2}L$ and so on.



The resonance frequencies of the cavity can be easily be found

$$\nu_m = \frac{c}{\lambda_m} = \frac{c}{m/2L} \Rightarrow \nu_m = m\frac{c}{2L}$$

For each frequency peak ν_m we have $\Delta\nu_{FWHM} = \frac{\Delta\nu_{FSR}}{\mathcal{F}} \Rightarrow$

$\Delta\nu_{FWHM} = \frac{\nu_m}{m\mathcal{F}}$. Calling $Q = m\mathcal{F}$, we obtain $Q = \frac{\nu_m}{\Delta\nu_{FWHM}}$

In the time domain we have a damped oscillating electric field with equation $E(t) = E_0\, e^{-\frac{t}{\tau_E}} \cos(\omega_0 t)$

In a time domain picture, the intensity of the EM field after a round trip. We wrote $I(2L) = I_0 R_1 R_2 (1-L_i)^2$

Calling t_1 the round trip time we can say

$$I(t_1) = I_0 R_1 R_2 (1-L_i)^2 \Rightarrow I(t_m) = I_0\left[R_1 R_2 (1-R_i)^2\right]^m.$$

Calling φ the number of photons inside the resonator, since $\varphi \propto I$ we can write that

$$\varphi(t_m) = \varphi_0\left[R_1 R_2 (1-L_i)^2\right]^m = \varphi_0 \exp\left[\ln\left(R_1 R_2 (1-L_i)^2\right)^m\right] \Rightarrow$$

$$\Rightarrow \varphi(t_m) = \varphi_0\, e^{m \ln[R_1 R_2 (1-L_i)^2]}$$

Remembering that we called $\ln(R_1) = -\gamma_1$; $\ln(R_2) = -\gamma_2$;

$\ln(1-L_i) = \gamma_i$ and $\gamma = \dfrac{\gamma_1+\gamma_2}{2} + \gamma_i$, we can write the number of photons inside the cavity at time t_m as

$$\varphi(t_m) = \varphi_0 \cdot e^{m\cdot 2\gamma}$$

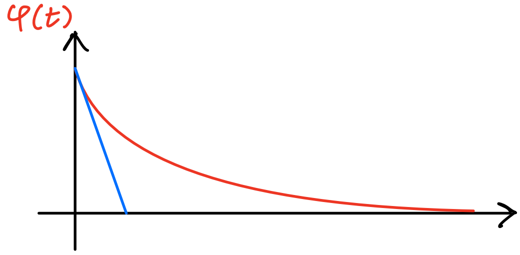
Since $t_m = m\,t_1 = m\dfrac{2L}{c} \Rightarrow m = \dfrac{t_m}{t_1} = t_m\dfrac{c}{2L}$, we obtain

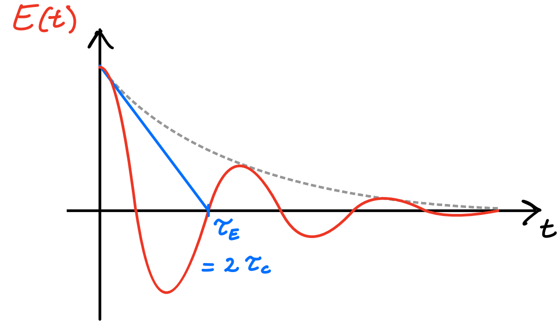
$$\varphi(t_m) = \varphi_0\, e^{\frac{\gamma c}{L}\cdot t_m}.$$

At a generic time t the number of photons behaves like

$\varphi(t) = \varphi_0\, e^{-\frac{t}{\tau_c}}$, where $\tau_c = \dfrac{L}{\gamma c} = \dfrac{\tau_{\text{single path}}}{\gamma}$ is called lifetime of a photon.

Calling U the total energy stored in the cavity, we know that $U \propto \varphi$, in particular $U(t) = h\nu_0\,\varphi(t) = h\nu_0\,\varphi_0\, e^{-t/\tau_c}$

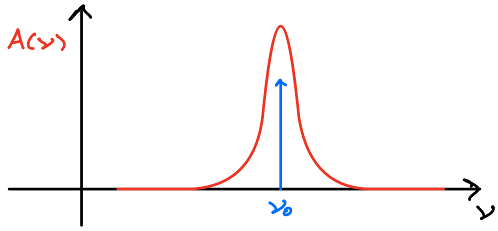




Since $E \propto \sqrt{I}$, we have that $\tau_E = 2\,\tau_c$ and

$$E(t) = E_0\, e^{-\frac{t}{\tau_E}} \cos(\omega_0 t)$$

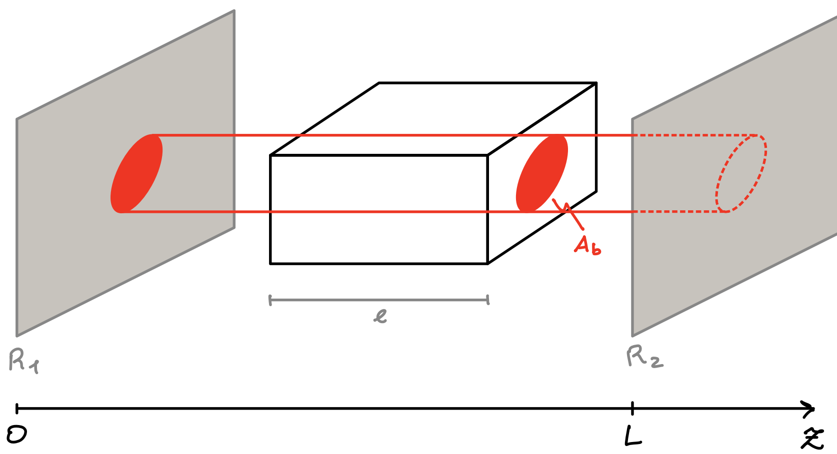
Its Fourier transform is a Lorentian function centered in ν_0: $\quad A(\nu) = \dfrac{A_0}{4\pi\tau_E^2(\nu-\nu_0)^2+1}$



Lasers rate equations

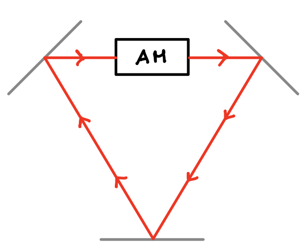
So far, we saw that the basic structure of a laser is made of a pumping system, an active material and an optical resonator.

We now want to find the number of photons inside the cavity φ and the output power, given a certain pumping rate. There are mainly two models: the space independent and the space dependent model. Considering the first one, we assume that the equations are valid in all the system. If the beam has an area A_b, its value does not change along z.



The two mirrors must be plane and the optical resonator must be unidirectional.

Hence, the Fabry-Perot resonator can not be used. A good resonator is the ring resonator, as sketched below.



The fourth assumption is that we are dealing with a 4-level system with very fast non-radiative decay

τ_{32} and τ_{10} as we saw before. We do not consider the spontaneous emission as a contribution the laser emission. Remember that we don't have contribution to the absorption.

$$\frac{dN_2}{dt} = \left(\frac{dN_2}{dt}\right)_{SE} + \left(\frac{dN_2}{dt}\right)_{spont} + \left(\frac{dN_2}{dt}\right)_{NR} = -\sigma\Phi N_2 - \left(\frac{1}{\tau_{sp}} + \frac{1}{\tau_{NR}}\right) N_2$$

We can define $\frac{1}{\tau} = \frac{1}{\tau_{sp}} + \frac{1}{\tau_{NR}}$

Let us consider the intensity after a round trip $I(2L)$.

$$I(2L) = I_0\, e^{2(\sigma N \ell - \gamma)} . \qquad \Delta I_{RT} = I(2L) - I_0 \quad (I \text{ variation})$$

$$\begin{cases} \Delta I_{RT} = I_0\left[e^{2(\sigma N\ell - \gamma)} - 1\right] \\ \Delta t_{RT} = \dfrac{2L_e}{c} \end{cases} \Rightarrow \quad \frac{\Delta I_{RT}}{\Delta t_{RT}} = I_0 \cdot \frac{c}{2L_e}\left[e^{2(\sigma N\ell - \gamma)} - 1\right]$$

Since ΔI_{RT} is very small, $e^{2(\sigma N\ell - \gamma)} - 1 \simeq 2(\sigma N \ell - \gamma)$

Eventually, we obtain the following equation

$$\frac{dI}{dt} = I_0 \frac{c}{L_e}\left[\sigma N \ell - \gamma\right] = \frac{\sigma N \ell c}{L_e} I - \frac{\gamma c}{L_e} I$$

$$I \propto \varphi \Rightarrow \frac{d\varphi}{dt} = \frac{\sigma N \ell c_0}{L_e}\varphi - \frac{\gamma c}{L_e}\varphi = \frac{\sigma N \ell c_0}{L_e}\varphi - \frac{1}{\tau_c}\varphi$$

This is the first rate equation.

The second equation takes into account the change of population:

$$\frac{dN}{dt} = R_p - B\varphi N - \frac{N}{\tau}$$

Calling V_a the volume of the active medium we have to impose that

$$\frac{\sigma N \ell c}{L_e}\varphi = B\varphi N V_a \implies B = \frac{\sigma \ell c}{V_a L_e} \quad \text{(stimulated em. constant)}$$

Eventually, the two rate equations can be written as

$$\begin{cases} \dfrac{dN}{dt} = R_p - B\varphi N - \dfrac{N}{\tau} \\ \dfrac{d\varphi}{dt} = B V_a N\varphi - \dfrac{\varphi}{\tau_c} \end{cases}$$

Let us now solve these equations in the continuous wave regime (stable operation), i.e. when the parameters do not change in time, in particular $\frac{dN}{dt} = 0$ and $\frac{d\varphi}{dt} = 0$.

$$\begin{cases} R_p - B\bar{\varphi}\bar{N} - \dfrac{\bar{N}}{\tau} = 0 \\ B V_a \bar{N}\bar{\varphi} - \dfrac{\bar{\varphi}}{\tau_c} = 0 \end{cases} \implies \left(B V_a \bar{N} - \frac{1}{\tau_c}\right)\bar{\varphi} = 0 \implies \left\langle \begin{array}{l} \bar{\varphi} = 0 \\ \bar{N} = \dfrac{1}{B V_a \tau_c} = \dfrac{\gamma}{\sigma \ell} \end{array} \right.$$

$\bar{N} = N_c = \frac{\gamma}{\sigma \ell}$ is called critical population inversion

The solution $\bar{\varphi} = 0$ is called below-threshold solution

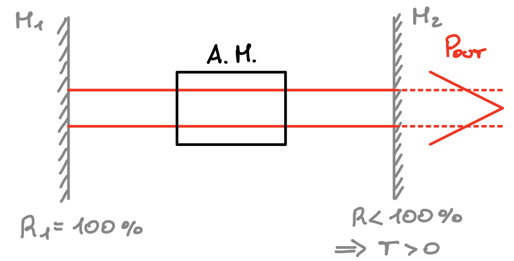
$$R_p - B\bar{\varphi}\bar{N} - \frac{\bar{N}}{\tau} = 0 \left\langle \begin{array}{l} \overset{\text{Below thresh.}}{} \; R_p - \dfrac{\bar{N}}{\tau} = 0 \implies \begin{cases} \bar{\varphi} = 0 \\ \bar{N} = R_p \tau \end{cases} \\[2ex] \underset{\text{Above thresh.}}{} \; R_p - B N_c \bar{\varphi} - \dfrac{N_c}{\tau} = 0 \implies \begin{cases} N = N_c = \dfrac{\gamma}{\sigma \ell} \\ \bar{\varphi} = \dfrac{1}{B N_c}\left[R_p - \dfrac{N_c}{\tau}\right] \end{cases} \end{array} \right.$$

$\varphi > 0$ if $R_p > R_{cp} = \frac{N_c}{\tau} = \frac{\gamma}{\sigma \ell \tau}$

Note that to have R_{cp} small we should have a large τ. Furthermore, $\bar{N} = R_p \tau$, hence the decay time should be long also to build up the population inversion.

<u>Lasers output power</u>

Let us study the output power in a system like the one sketched below. The output power can be written as



$$P_{OUT} = \left.\frac{d\varphi}{dt}\right|_{M_2} \cdot h\nu_0 = \frac{\overline{\varphi}}{\tau_c} \cdot \frac{\gamma_2}{2\gamma} \cdot h\nu_0 ,$$

hence it is the product of the total photons lost by the cavity, times the fraction due to a pass through M_2 times the energy of a photon. Remember that

$$\overline{\varphi} = \frac{1}{BN_c}(R_p - R_{cp}); \quad B = \frac{\sigma \ell c}{V_a L_a}; \quad N_c = \frac{\gamma}{\sigma \ell} \Rightarrow BN_c = \frac{\gamma c}{V_a L_a} = \frac{1}{V_a \tau_c}$$

$\Rightarrow \overline{\varphi} = V_a \tau_c (R_p - R_{cp})$. Multiplying by $\frac{R_{cp}}{R_{cp}}$ we obtain

$$\overline{\varphi} = \left(\frac{R_p}{R_{cp}} - 1\right) V_a \tau_c R_{cp} = \left(\frac{R_p}{R_{cp}} - 1\right) V_a \tau_c \frac{\gamma}{\sigma \ell \tau}$$

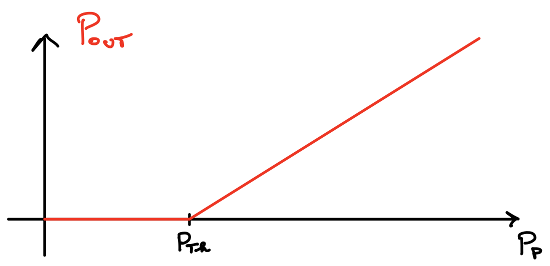
We know that $R_p \propto P_p$ and $R_{cp} = P_{cp}$, thus $\frac{R_p}{R_{cp}} = \frac{P_p}{P_{Th}}$, so:

$$P_{OUT} = h\nu_0 \frac{\gamma_2}{2\cancel{\gamma}} \cdot \frac{1}{\cancel{\tau_c}} \frac{\cancel{\gamma}}{\sigma \ell \tau} V_a \cancel{\tau_c} \left(\frac{P_p}{P_{Th}} - 1\right) = \frac{\gamma_2}{2} I_s A_b \left(\frac{P_p}{P_{Th}} - 1\right)$$

Where we put $I_s = \frac{h\nu_0}{\sigma \tau}$ (saturation intensity) and $A_b = \frac{V_a}{\ell}$

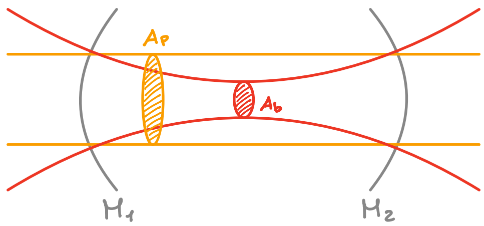
Their product is called saturation power.

The figure aside shows the plot of P_{OUT} as a function of the pump power. The derivative of P_{OUT}, $\frac{dP_{OUT}}{dP_p} = \eta_s$, is called slope



efficiency of the laser: $\eta_s = \frac{\gamma_2}{2} I_s A_b \cdot \frac{1}{P_{Th}}$

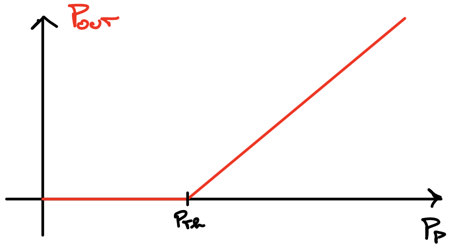
$$P_{Th} = \frac{R_{cp} V_p h\nu_{mp}}{\eta_p} \Rightarrow \eta_s = \eta_p \frac{\gamma_2}{2\gamma} \frac{h\nu_0}{h\nu_{mp}} \frac{A_b}{A_p} = \eta_p \eta_c \eta_q \eta_t$$



η_t is the ratio between the pump beam area and the laser beam. This ratio can be non-unitary in some cases, for example in the case of curved mirrors in the figure above.

Remember that mirrors should be HR at laser wavelengths and HT at pump wavelength.

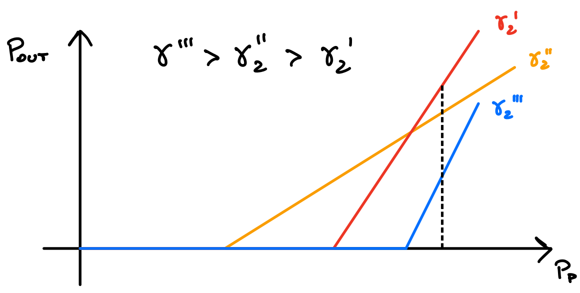
We saw that the plot of the output power as a function of the pump power is the one shown aside.

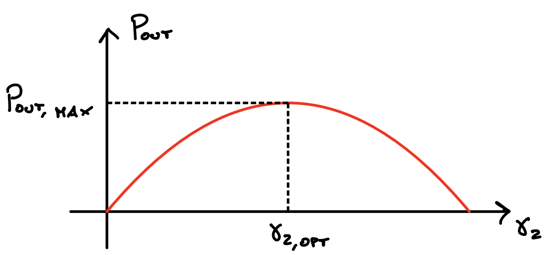


$$P_{OUT} = \frac{\gamma_2}{2} I_s A_b \left(\frac{P_P}{P_{TH}} - 1 \right),$$ hence $P_{OUT} \propto \gamma_2$.

There are two issues: increasing γ_2, γ increases, thus the losses increase; increasing γ_2, γ increases and R_{cp} increases, since $R_{cp} = \frac{N_c}{\tau} = \frac{\gamma}{\sigma l \tau}$, so we increase the threshold pump power.

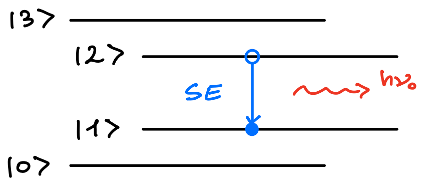
We eventually reach a point in which the increasing of the threshold is too high. Hence, the qualitative plot of P_{OUT} as a function of γ_2 for a fixed P_{TH} is the following, with a maximum in $\gamma_{2,OPT}$.





Emission frequency in steady state

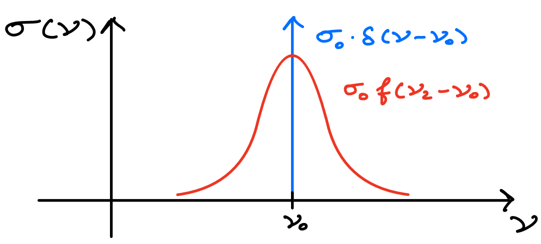
Let us now study in a qualitative way the emission frequency ν_0 for a laser in continuous wave. A formal treatment will be given later.



$$h\nu_0 = E_2 - E_1 \Rightarrow \nu_0 = \frac{E_2 - E_1}{h}$$

Actually, the two energy levels are not so well defined, hence we usually don't have a precise monochromatic emission frequency ν_0. Remember that

$$\left(\frac{dN_2}{dt}\right)_{SE} = -W_{21} N_2 \,,$$ where $W_{21} = \Phi(\nu)\cdot\sigma_{21}(\nu)$, so it depends on the frequency. Ideally σ_{21} in frequency is a Dirac delta.

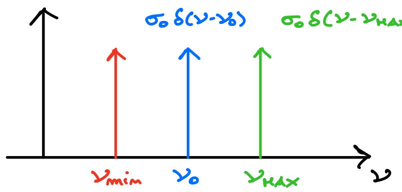


This would happen only if all the molecules have exactly the same energy distribution. In the real world there's always uncertainty in the interaction energy.

The laser levels |1> and |2> are not well defined but have an uncertainty ΔE_1 and ΔE_2. So, the stimulated emission can take place between $h\nu_{min} = E_2 - E_1$ and $h\nu_{MAX} = E_2 + \Delta E_2 - (E_1 + \Delta E_1)$.

Now $\sigma_{21} = \sigma_0 f(\nu-\nu_0)$, where f is a distribution function with maximum equal to 1.

We have homogeneous broadening if each atom can undergo stimulated emission at different frequency and $\sigma_{21}(\nu)$ is the same for every atom. For the homogeneous broadening $f(\nu)$ is a Lorentian function.
Instead, we have inhomogeneous broadening if the radiation-matter interaction takes place in an ideal way (no spreading in the energy levels), but the atoms do not have exactly the same energy distribution. In this case each atom has its S.E. cross section.

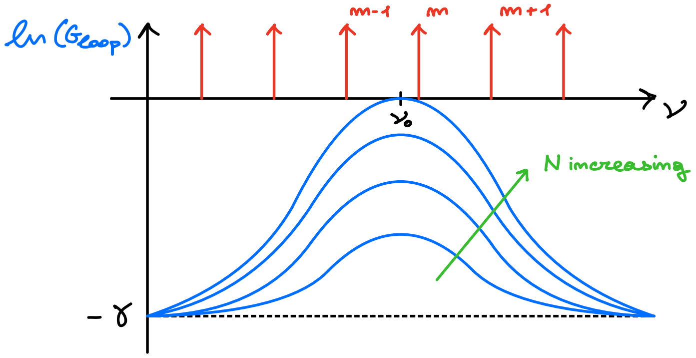


We obtain a distribution of Dirac deltas in frequency.
Taking the ensamble average of the SE cross section $\langle \sigma_{SE} \rangle$ we obtain a Gaussian distribution given by $\langle \sigma_{SE} \rangle = \sigma_0 f^*(\nu-\nu_0)$. $\Delta\nu_0^*$ is the FWHM of the inhomogeneous broadened cross section and f^* is a Gaussian function.
The homogeneous $\Delta\nu_0$ is of several MHz, while in the inhomogeneous case the $\Delta\nu_0^*$ goes up to hundreds of GHz or even THz.
Remember that the monochromaticity of a laser is given by the active material amplification and the optical resonator.
We saw that the G_{loop} into an unidirectional resonator is

$$G_{loop} = \frac{I(2L)}{I_0} = e^{\sigma N \ell - \gamma}.$$

For the Barkhausen condition $G_{loop} = 1 \Rightarrow \sigma N \ell - \gamma = 0$

The gain of the material per unit length is $g = \sigma N$, so $g = g(\nu)$ and we impose $g(\nu)\ell - \gamma = 0$. So, the Barkhausen condition graphical representation as a function of ν becomes the following.



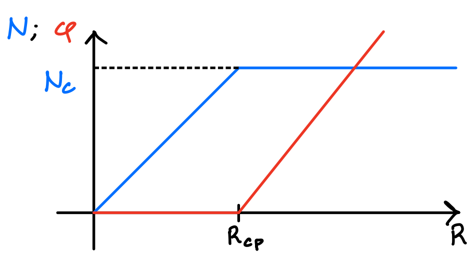
$g(\nu) = N\sigma_0 f(\nu - \nu_0)$,

hence the gain increases linearly with N.

In the blue plots, ν_0 satisfies the condition, but it does not precisely correspond to a resonant frequency.

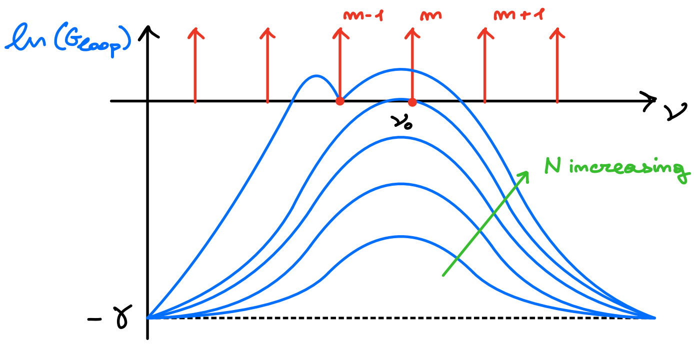
Hence, the gain has to be further increased until the frequency $\nu_m = m\Delta\nu_{FSR}$ starts lasing. ν_m is the resonant frequency of the cavity closest to the peak ν_0 of the gain.

When $g = N_c \sigma(\nu - \nu_0) = \dfrac{\gamma}{\sigma(\nu_L)\ell}$, the gain stops growing since N is fixed at N_c and the gain saturates. This prevents the gain to further increase activating other frequencies.



The saturation of N, shown aside, happens when photon number starts to grow inside the cavity.

In case of inhomogeneously broadened active material an effect called spectral hole burning happens.



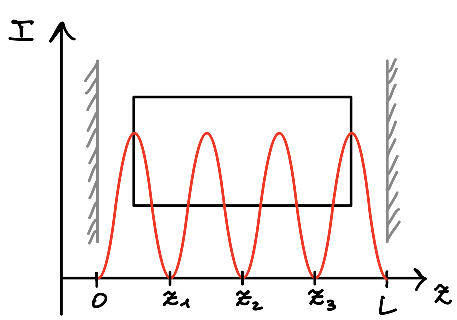
In this case some atoms are able to emit only at particular frequencies, so we reach the population inversion only for some emitters, but all the other atoms can keep growing the gain, turning on as many modes we want. We only saturate the population of some particular emission frequencies.

The number of oscillating mode is given by $m = \frac{\Delta\nu_0^*}{\Delta\nu_{FSR}}$

If $\Delta\nu_0^*$ is large (for example in the sapphire laser) with small $\Delta\nu_{FSR}$ we can even reach $m = 10^5 \div 10^6$.

Another effect, that takes places also in homogeneous broading material is the spatial hole burning. This is the main cause of multimodal operation in HB materials.

Let us consider an optical bidirectional resonator. The

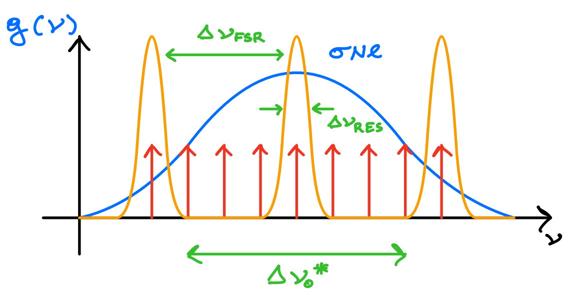


optical intensity has the shape shown aside, with frequency ν_L. In the points of the material with $I = 0$ there are no photons. In those points we have no intensity for ν_L, so another lasing frequency ν_L' can come up in z_1, z_2 and z_3 since we can not have saturation of the gain.

Emission wavelength selection and tuning

In case of H.B., to prevent spatial hole burning we can adopt a unidirectional cavity design, for example a ring resonator. We may then use an optical diode to allow light to circulate only in one way.

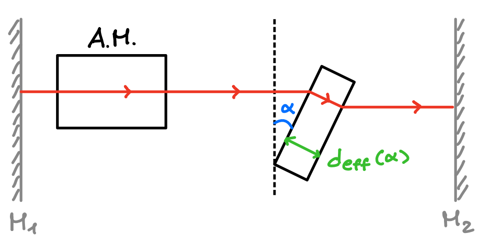
For I.B. active media we have spectral hole burning. We have low losses for a single cavity mode and high losses for all the other modes. This can be made using a Fabry-Perot etalon.



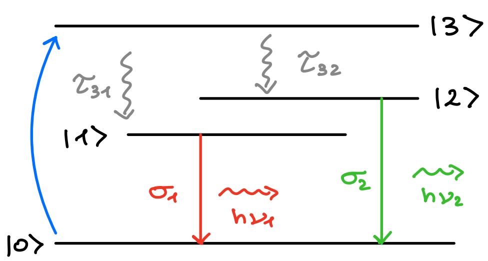
We design the structure in order to have a peak in the maximum of $g(\nu)$ and the two adjacent peaks of the etalon out of $\Delta\nu_0^*$. In this way the etalon introduces too high losses for the other frequencies.

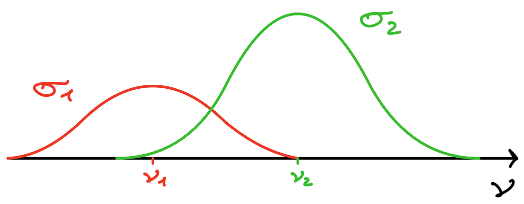
The structure scheme is shown below. The two conditions to be satisfied are:

- $\Delta\nu_{FSR} = \dfrac{c}{2d_{eff}} > \dfrac{\Delta\nu_0^*}{2}$
- $\Delta\nu_{RES} < \dfrac{c}{2L_e} = \dfrac{\Delta\nu_{FSR}}{\mathcal{F}}$



Let us consider a three level system, but with two possible emitting levels, $|1\rangle$ and $|2\rangle$, with stimulated emission σ_1 and σ_2

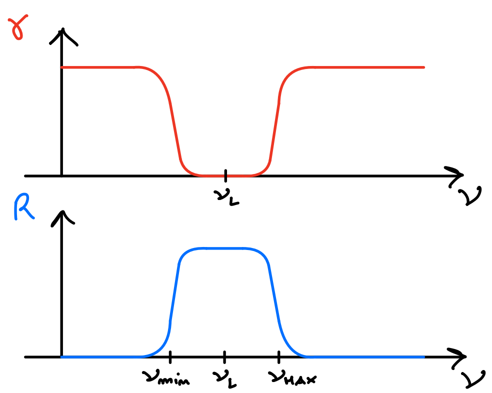




If $\tau_{32} > \tau_{31}$, emission over σ_1 at ν_1 will be favoured; if instead $\sigma_2 > \sigma_1$, emission over σ_2 is favoured.

By introducing a mechanism of wavelength selective losses it's possible to prevent laser action at one frequency and tune the emission frequency.

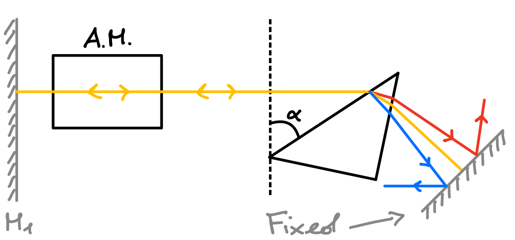
To do this, the simplest idea is to design HR mirrors at the desired wavelength, in order to have high losses (high γ)



for the undesired frequencies and low losses at ν_L.

The issue with this structure is that the losses are not tunable.

To solve this problem we could introduce a prism inside the cavity. For example, in the figure below, only the yellow beam

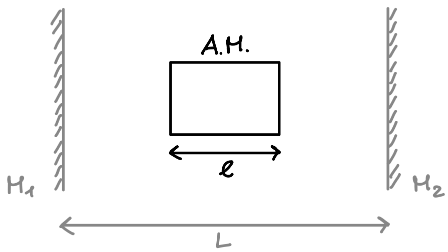


is alligned to the cavity; the mirror is fixed, while the angle α is tunable. These are called alignment dependent losses. Small changes of α allow to select λ with good cavity alignment, chosing the cavity mode.

Another method is the so-called diffraction grating.

Sources of laser frequency noise

Let us consider a general laser cavity of length L. The optical length is given by



$$L_e = L + \ell(n-1)$$

L, ℓ and n can all change in time due to environmental conditions.

The refractive index is a fixed value plus a variation affected by the temperature or by the pump power. ℓ can change with temperature; we can usually write the variation as $\delta\ell = \ell_0 \cdot \alpha(T-T_0)$, so it is linear. α is of the order of $10^{-6}\,K^{-1}$ for most materials, for ULE glasses α can be decreased down to $10^{-9}\,K^{-1}$. L can also change and typically $L = L_0 + \delta L(P_a)$, where P_a is the acoustical noise pressure. Thus:

$$\begin{cases} n = n_0 + \delta n(T, P_p) \\ \ell = \ell_0 + \delta\ell(T) = \ell_0(1+\alpha(T-T_0)) \\ L = L_0 + \delta L(P_a) \end{cases}$$

T slowly varies in time and we call it temperature drift. P_a, instead, changes faster in time. The time constant for acoustical noise is called jitter of the cavity and its frequency is typically in the range $1\,Hz \div 100\,KHz$. The P_p behaviour in time varies with the type of laser; for example, it is very slow for solid state lasers and very high for laser diodes (up to tens of MHz).

Hence, $L_e = L_e + \delta L_e(T(t), P_a(t), P_p(t))$. The frequency of the laser is given by

$$\nu_L = m\frac{c}{2L_e} \implies \delta\nu_L = \nu_L \frac{\delta L_e}{L_e} = \delta\nu_L(t)$$

We can divide L_e into a sum of drift contributions and jitter contributions: $\delta L_e(t) = \delta L_{e_D}(t) + \delta L_{e_J}(t)$, hence

$\delta\nu_L = \delta\nu_{L,J}(t) + \delta\nu_{L,D}(t)$.

The emission frequency of the laser can be written as

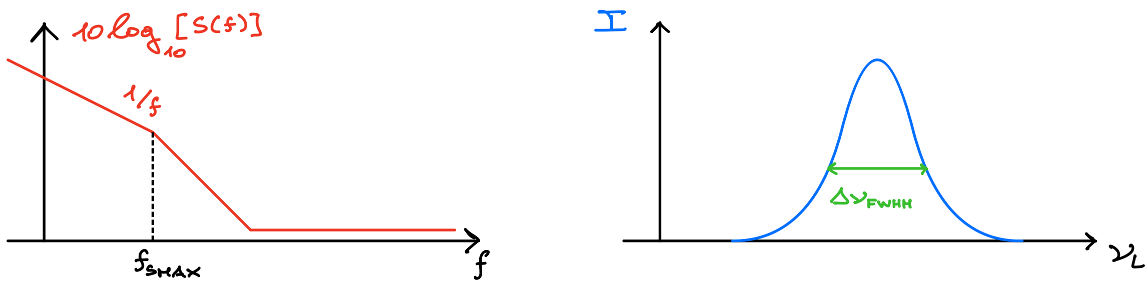
$$\nu_L(t) = \overline{\nu}_L + \Delta\nu_{L,J}(t) + \Delta\nu_{L,D}(t),$$

where $\overline{\nu}_L$ is a fixed value not affected by noise.

Drift contributions can be usually considered $1/f$ noise.

$\Delta\nu_J$ can bring to $10\text{ Hz} \div 10\text{ MHz}$ optical frequency variations, while $\Delta\nu_D$ can bring up to several GHz frequency variations.

The Bode diagram of the frequency noise has the following qualitative shape



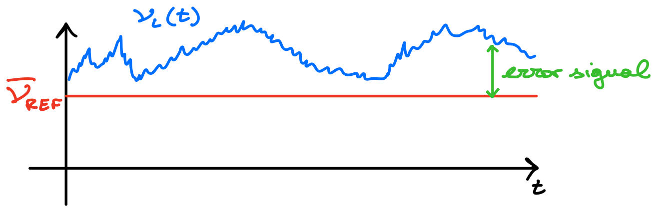
On the right, the emission linewidth is sketched, where $\Delta\nu_{FWHM}$ can go from few KHz to several MHz.

Most lasers applications rely on the fact that the emission frequency is very stable. Thus, we have to study a way to stabilize the emission frequency.

Stabilization of the emission frequency

In order to stabilize the frequency we have to compensate with an external action the fluctuations. The elements needed are:

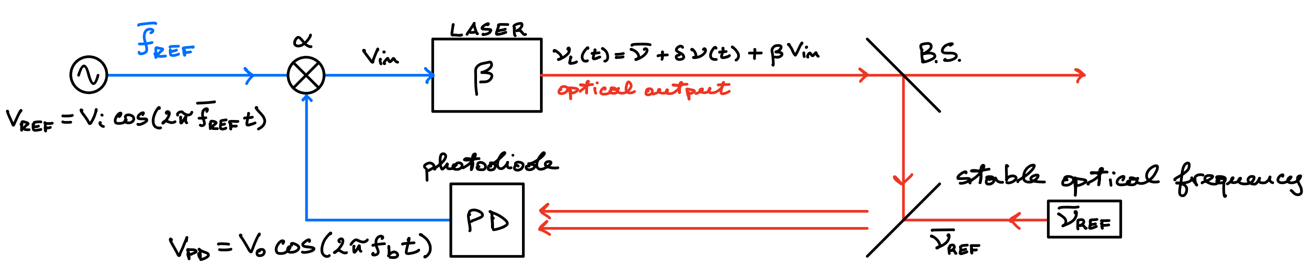
1) Actuators that change L_e to compensate T, P_a and P_p fluctuations. We could use a piezo actuator on the mirrors or a Peltier cell in the active material to control the temperature.
2) Stable frequency references (optical and RF) to compare the output frequency.
3) Negative feedback loop applied to the system.



The error signal in figure has to be negatively fed back to the actuators.

For the RF reference we can use a Quartz or a Rb oscillator (10^{-12} relative stability) or a Cs clock (10^{-16} relative stability per second).

The block diagram of the system is the following.



Optical signals are represented in red, while voltage signals are represented in blue.

βV_{in} is the ν tuning thanks to the actuator. f_b is the beatnote frequency and usually falls into the RF region.

Calling α the gain of the mixer, we have that

$$\begin{cases} \varepsilon = V_{MIX,OUT} = V_{in} = \alpha \left[f_b(t) - f_{REF} \right] \\ f_b = \bar{\nu} + \beta V_{in} + \delta\nu - \bar{\nu}_{REF} \end{cases} \Rightarrow \varepsilon = \alpha \left[\bar{\nu} + \beta\varepsilon + \delta\nu - \bar{\nu}_{REF} - \bar{f}_{REF} \right]$$

$$\Rightarrow \varepsilon = \frac{\alpha \left[\bar{\nu} + \delta\nu - \bar{\nu}_{REF} - \bar{f}_{REF} \right]}{1 - \alpha\beta}, \quad \text{where } \alpha\beta = G_{loop}$$

Hence, the laser output frequency will be given by

$$\nu_L(t) = \bar{\nu} + \beta \frac{\alpha \left[\bar{\nu} + \delta\nu - \bar{\nu}_{REF} - \bar{f}_{REF} \right]}{1 - \alpha\beta} + \delta\nu(t) \Rightarrow$$

$$\nu_L(t) = \bar{\nu} + \frac{\alpha\beta}{1 - \alpha\beta} \left[\bar{\nu} + \delta\nu - \bar{\nu}_{REF} - \bar{f}_{REF} \right] + \delta\nu(t)$$

The gain of the system is $G_{ID} = \dfrac{\alpha\beta}{1 - \alpha\beta}$. For $\alpha\beta = G_{loop} \gg 1$, we obtain $G_{ID} = -1$. Hence, for $\alpha\beta \rightarrow +\infty$ we have

$$\nu_L(t) = \bar{\nu} - \bar{\nu} - \cancel{\delta\nu(t)} + \bar{\nu}_{REF} + \bar{f}_{REF} + \cancel{\delta\nu(t)} = \bar{\nu}_{REF} + \bar{f}_{REF}$$

Both $\bar{\nu}_{REF}$ and $\bar{f}_{REF}$ are very stable.

So, ideally, we can totally remove the noise contributions.

Of course the former statement holds until the G_{loop} crosses the ideal gain. Calling f_P the frequency of the dominant pole, the maximum noise frequency we can compensate is

$$\alpha\beta \cdot f_P = |G_{ID}| \cdot f_{MAX} \Rightarrow f_{MAX} = \alpha\beta f_P .$$

Transient laser operation

To study the transient operation we adopt a perturbative approach. The rate equations are:

$$\begin{cases} \dfrac{dN}{dt} = R_p - B\varphi N - \dfrac{N}{\tau} = f(N,\varphi) \\ \dfrac{d\varphi}{dt} = BV_a N\varphi - \dfrac{\varphi}{\tau_c} = g(N,\varphi) \end{cases}$$

Consider a laser that works in CW with a population $\bar{N}$ and a number of photons $\bar{\varphi}$. Let us apply a small perturbation for N and φ: $N(t) = \bar{N} + \delta N(t)$ and $\varphi(t) = \bar{\varphi} + \delta\varphi(t)$, with $\delta N \ll \bar{N}$ and $\delta\varphi \ll \bar{\varphi}$.

Note: from now on we will use the Newton notation for the time derivative.

Substituting N(t) and $\varphi(t)$ in the rate equations, observing that $\dot{\bar{N}} = 0$ and $\dot{\bar{\varphi}} = 0$, and neglecting the second order terms $\delta\varphi \cdot \delta N$, we obtain the following linear equations:

$$\begin{cases} \delta\dot{N} = \left.\dfrac{\partial f(N,\varphi)}{\partial N}\right|_{\bar{N},\bar{\varphi}} \delta N + \left.\dfrac{\partial f}{\partial \varphi}\right|_{\bar{N},\bar{\varphi}} \delta\varphi \\ \delta\dot{\varphi} = \left.\dfrac{\partial g(N,\varphi)}{\partial N}\right|_{\bar{N},\bar{\varphi}} \delta N + \left.\dfrac{\partial g}{\partial \varphi}\right|_{\bar{N},\bar{\varphi}} \delta\varphi \end{cases}$$

In matrix form we can write:

$$\begin{bmatrix} \delta\dot{N} \\ \delta\dot{\varphi} \end{bmatrix} = \begin{bmatrix} -(B\bar{\varphi} + 1/\tau) & -B\bar{N} \\ BV_a\bar{\varphi} & BV_a\bar{N} - 1/\tau_c \end{bmatrix} \begin{bmatrix} \delta N \\ \delta\varphi \end{bmatrix}$$

To find the solution we have to find the eigenvalues λ_1, λ_2

of the matrix. In order to have a stable system we should have $\lambda_1, \lambda_2 < 0$. If they are complex conjugated we have an oscillation of N and φ in time.

We can have two different regime conditions: above and below threshold.

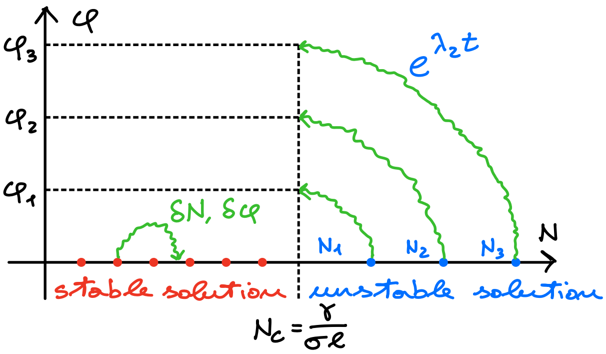
1) Laser below threshold: $\overline{\varphi} = 0$. The rate equations become

$$\begin{bmatrix} \delta\dot{N} \\ \delta\dot{\varphi} \end{bmatrix} \begin{bmatrix} 1/\tau & -B\overline{N} \\ 0 & BV_a\overline{N} - 1/\tau_c \end{bmatrix} \begin{bmatrix} \delta N \\ \delta\varphi \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda + 1/\tau & B\overline{N} \\ 0 & \lambda - (BV_a\overline{N} + 1/\tau_c) \end{vmatrix}.$$ It's a triangular matrix.

Hence: $\lambda_1 = -\frac{1}{\tau}$; $\lambda_2 = BV_a\overline{N} - \frac{1}{\tau_c}$. $\begin{cases} \overline{N} > \frac{1}{BV_a\tau_c} \Rightarrow \lambda_2 > 0 \\ \overline{N} > \frac{1}{BV_a\tau_c} \Rightarrow \lambda_2 < 0 \end{cases}$

So, we can draw the following plot for below threshold regime.



When $N > N_c$ we have an unstable solution: a perturbation causes a reduction of N and an increasing of φ, proportional to $\overline{N}$.

2) Laser above threshold: $\overline{\varphi} \neq 0$. The linearized system is

$$\begin{bmatrix} \delta\dot{N} \\ \delta\dot{\varphi} \end{bmatrix} = \begin{bmatrix} -(B\overline{\varphi} + 1/\tau) & -B\overline{N} \\ BV_a\overline{\varphi} & BV_a\overline{N} - 1/\tau_c \end{bmatrix} \begin{bmatrix} \delta N \\ \delta\varphi \end{bmatrix}$$

In this case we will always have $\bar{N} = N_c = \frac{\gamma}{\sigma\ell} = \frac{1}{BV_a\tau_c}$

$$|\lambda I - A| = \begin{vmatrix} \lambda + B\bar{\varphi} + 1/\tau & B\bar{N} \\ -BV_a\bar{\varphi} & \lambda \end{vmatrix} = \lambda^2 + \left(B\bar{\varphi} + \frac{1}{\tau}\right)\lambda - B\bar{N}V_a\bar{\varphi} = 0$$

$$\bar{\varphi} = \frac{1}{BN_c}\left[R_p - R_{pc}\right] = \frac{R_p}{BN_c}\left[\frac{R_p}{R_{pc}} - 1\right] = \frac{1}{B\tau}(x-1), \quad \text{where } x = \frac{R_p}{R_{pc}}$$

$$\Rightarrow \lambda^2 + \left(B\bar{\varphi} + \frac{1}{\tau}\right)\lambda - BV_a\bar{N}\bar{\varphi} = 0; \quad \bar{N} = \frac{1}{BV_a\tau_c}, \quad \bar{\varphi} = \frac{1}{B\tau}(x-1)$$

So, $B\bar{\varphi} + \frac{1}{\tau} = B \cdot \frac{1}{B\tau}(x-1) + \frac{1}{\tau} = \frac{x}{\tau} = \frac{2}{t_o}$, where $t_o = \frac{2\tau}{x}$ and

$BV_a\bar{N}\bar{\varphi} = BV_a \cdot \frac{1}{BV_a\tau_c} \cdot \frac{1}{B\tau}(x-1) = \frac{1}{B\tau}(x-1) = \omega_o^2$, where

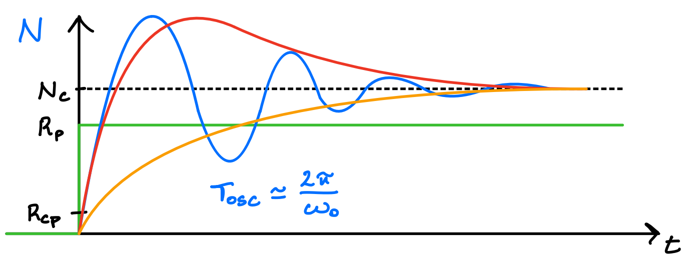
$\omega_o = \sqrt{\frac{1}{\tau\tau_c}(x-1)}$ is called natural relaxation frequency.

$$\lambda^2 + \frac{2}{t_o}\lambda + \omega_o^2 = 0 \Rightarrow \lambda_{1,2} = \frac{1}{t_o} \pm j\sqrt{\omega_o^2 - \frac{1}{t_o^2}} = -\frac{1}{t_o}\left[1 \pm \sqrt{\omega_o^2 t_o^2 - 1}\right]$$

$\mathrm{Re}\{\lambda_{1,2}\} < 0$ always, hence the system is always stable with some oscillations due to the immaginary part if $\omega_o^2 t_o^2 > 1$. We can write

$$\omega_o^2 t_o^2 = \frac{4\tau}{\tau_c} \cdot \frac{x-1}{x^2} \simeq \frac{4\tau}{\tau_c} \cdot \frac{1}{x}, \quad \text{since } x \gg 1.$$

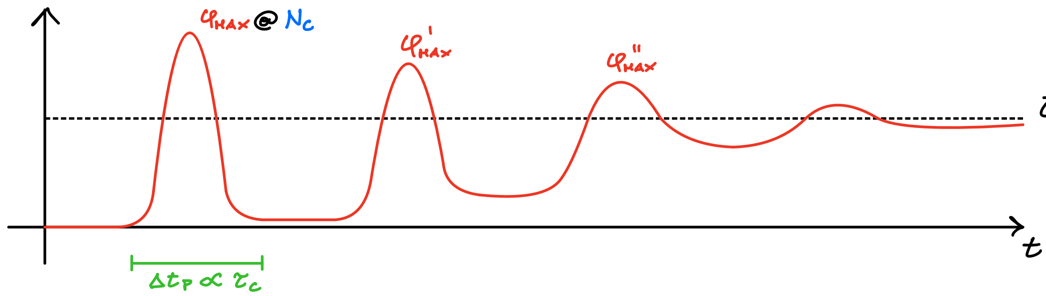
$\frac{4\tau}{\tau_c} \cdot \frac{1}{x} \gg 1$ if $\tau \gg \tau_c$. The photon number $\bar{\varphi}(t)$ oscillates as $\bar{\varphi}(t) \propto e^{-\frac{t}{t_o}} \cdot e^{j\omega_o t}$ (under-damped solution)



If $\omega_o^2 t^2 > 1$ we will have the solution in blue; if instead $\omega_o^2 t^2 < 1$

we obtain an overdamped solution (the red and the orange in the figure).

The dynamic of the photon number can be represented as follows:

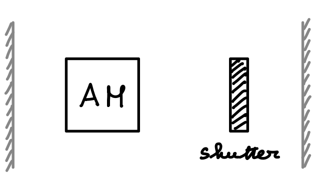


It has some pulses with maximums in correspondance with N_c and it arrives at steady state at $\bar{\varphi}$

Q-switching

From the previous plot we can say that every time the laser is turned on we have a pulse that has a duration of about τ_c. We can exploit this property to let the laser work in pulsed regime.

Let us put a shutter into the cavity in order to prevent the photons to recirculate inside it.



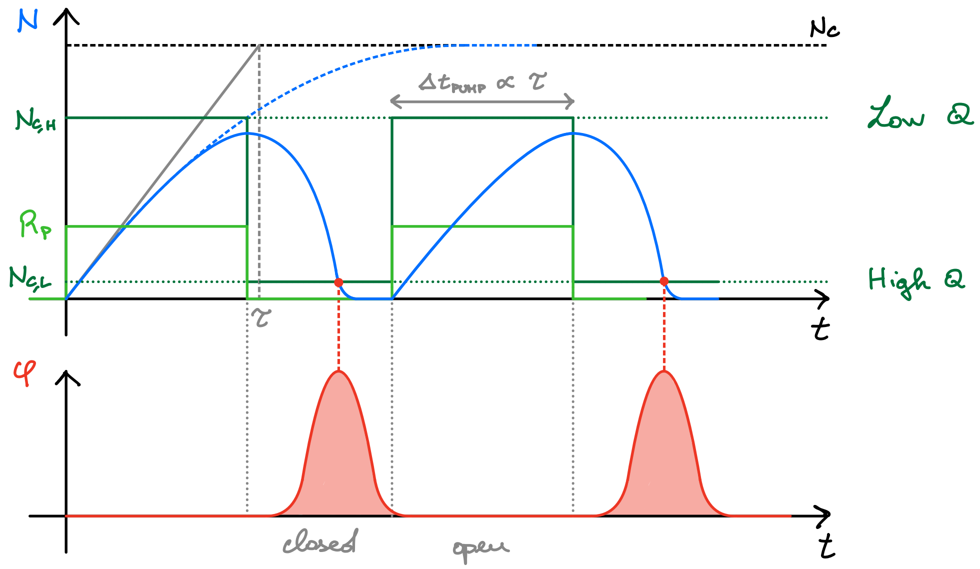
When the shutter is closed we have high losses and we can call $\gamma = \gamma_{HIGH}$.

We artificially prevent the onset of the laser action.

When the shutter is open we call $\gamma = \gamma_{LOW}$. The critical population inversion in the two cases is:

$$N_{c,L} = \frac{\gamma_{LOW}}{\sigma \tau} \ll N_{c,H} = \frac{\gamma_{HIGH}}{\sigma \tau}$$

The name Q-switching is due to the fact that a high/low γ means a low/high quality factor, hence we have a switching of Q.



Typical range of pulse repetition frequency is

$f_p = \frac{1}{T_p} = 1\,KHz \div 100\,KHz$ and peak power up to hundreds of KW.

There are several Q switching methods, that can be divided into active and passive switches.

Main active switches are:

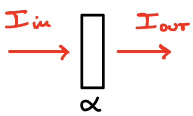
- Rotating prism (alignment-dependent losses).

  It is easy to implement and gives low values of γ_{LOW}, but it is not very fast since it is limited by the motor speed.

- Intra-cavity optical modulator. It can be electro-optical (EOM) or acousto-optical.

  They are very fast, but have a high cost and γ_{LOW} is higher than the rotating prism.

Passive Q-switches exploit saturable absorber materials. They are called SESAM (Semiconductor Saturable Absorber Modulators). The absorption coefficient depends on the intensity as follows



$$\alpha = \sigma_{ABS} N = \frac{\alpha_0}{1 + I_{in}/I_s} \implies \alpha_{MIN} = \frac{\alpha_0}{1 + I_{PEAK}/I_s}$$

The advantages are that we don't need to synchronize a shutter and it leads to shorter pulse durations.

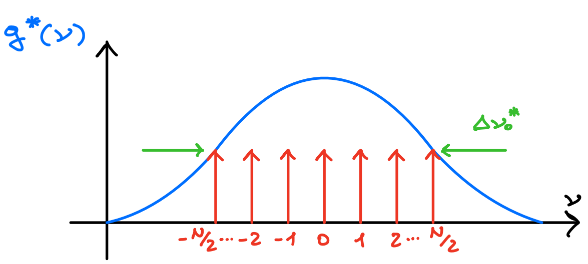
They introduce high losses in the low-γ condition.

Mode locking

The advantages of the Mode-locking w.r.t. Q-switching are

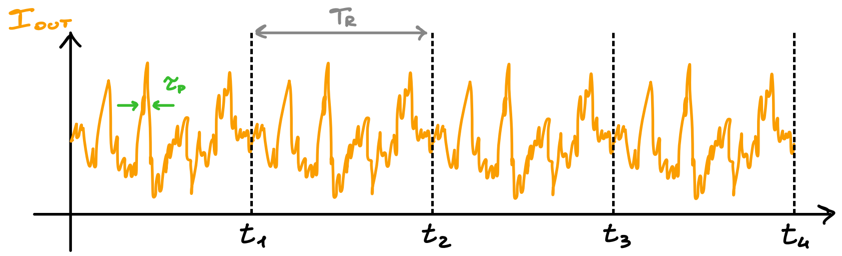
1) Higher pulse repetition rate (20 MHz - 1 GHz).
2) Pulses are coherent.
3) Peak power up to 100 KW.

For Mode locking we need a broadband gain spectrum material inhomogeneously broadened, in order to have spectral hole burning.



Remember that the gain spectrum $g^*(\nu)$ is a Gaussian. The number of oscillating modes is $N = \dfrac{\Delta\nu_0^*}{\Delta\nu_c} \simeq 10^5 \div 10^6$

Let us assume to operate this laser without any mean to select a single mode. The output intensity of this laser will be very noisy, but it will also be periodic with a



period

$$T_R = \frac{2L_e}{c}, \quad \text{so}$$

$$t_m = m \cdot \frac{2L_e}{c}$$

If we look at the duration of each peak we find that they all have a duration of about $\tau_p = (\Delta\nu_0^*)^{-1}$.

This happens because of the coherent superposition of all the oscillating modes. Calling the modes $\left(-\frac{N}{2}, \ldots, \frac{N}{2}\right)$ like in the spectrum plot above we can write, in a particular space position

$$E_0(t) = E_0\, e^{j(\omega_0 t + \varphi_0)} \; ; \qquad E_\ell = E_0\, e^{j[(\omega_0 + \ell\Delta\omega)t + \varphi_\ell]}$$

where $\Delta\omega$ is the distance between a mode and the following one, hence $\Delta\omega = 2\pi \frac{c}{2L_e}$

The phase φ_ℓ is randomic unless we exploit a way to relate the phases of each mode with the others.

Since the spectrum is discrete, the output time behaviour must be periodic in time. Thanks to the Fourier transform properties we know that the frequency repetition rate is equal to $f_R = \frac{\Delta\omega}{2\pi} = \frac{c}{2L_e}$.

Since $g^*(\nu) = \mathcal{F}[E(t)] = \mathcal{F}\left[\sum_{\ell=-N/2}^{N/2} E_\ell(t)\right]$, the minimum width of the peaks must be $\tau_p = \frac{1}{\Delta\nu_0^*}$.

Let us assume that we are able to build a relationship between the phases of the different modes, for example $\varphi_{\ell+1} - \varphi_\ell = \bar{\varphi}$ (constant). By convention we can fix $\varphi_0 = 0$. Thus, $\varphi_1 = -\varphi_1 = \bar{\varphi}$ and $\varphi_\ell = -\varphi_\ell = \ell\bar{\varphi}$. In this way:

$$E_\ell(t) = E_0 e^{j[(\omega_0 + \ell\Delta\omega)t + \ell\bar{\varphi}]} \Rightarrow$$

$$E_{TOT}(t) = \sum_{\ell=-N/2}^{N/2} \left\{ E_0 e^{j\omega_0 t} e^{j\ell(\Delta\omega \cdot t + \bar{\varphi})} \right\} = E_0 e^{j\omega_0 t} \sum_{\ell=-N/2}^{N/2} e^{j\ell(\Delta\omega t + \bar{\varphi})}.$$

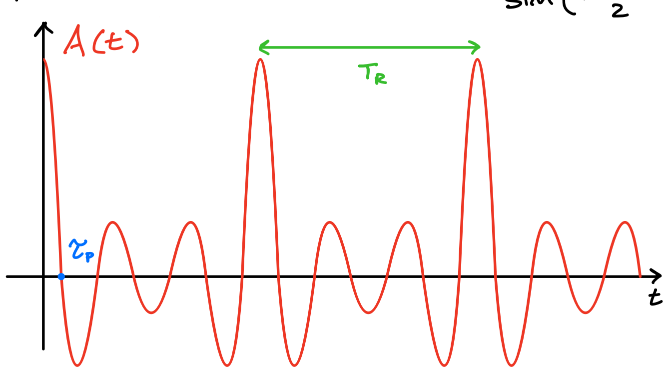
Calling $\Delta\omega t + \bar{\varphi} = \Delta\omega t' \Rightarrow t' = t + \frac{\bar{\varphi}}{\Delta\omega}$, hence:

$$E_{TOT}(t) = \underbrace{E_0 e^{j\omega_0 t}}_{\text{carrier}} \underbrace{\sum_{\ell=-N/2}^{N/2} e^{j\ell \cdot \Delta\omega t'}}_{\text{envelope}} = E_0 e^{j\omega_0 t} \cdot A(t)$$

Note that $A(t)$ is a geometric series, so it converges to

$$A(t') = \sum_{\ell=-N/2}^{N/2} e^{j\ell \cdot \Delta\omega t'} = \frac{\sin\left(\frac{N\Delta\omega t'}{2}\right)}{\sin\left(\frac{\Delta\omega \cdot t'}{2}\right)}.$$

Assuming $t \simeq t'$, we obtain the final expression for the pulse envelope $A(t) = \dfrac{\sin\left(\frac{N\Delta\omega t'}{2}\right)}{\sin\left(\frac{\Delta\omega \cdot t'}{2}\right)}$, and $I(t) = A^2(t)$



We have the maxima for

$$\sin\left(\frac{\Delta\omega \cdot t_M}{2}\right) = 0 \Rightarrow \frac{\Delta\omega \cdot t_M}{2} = m\pi$$

$$\Rightarrow t_m = m\frac{2\pi}{\Delta\omega} = m \cdot \frac{1}{\Delta\nu_c},$$ so

the repetition period becomes $T_R = \dfrac{1}{\Delta\nu_c} \Rightarrow f_R = \dfrac{1}{T_R} = \Delta\nu_c$

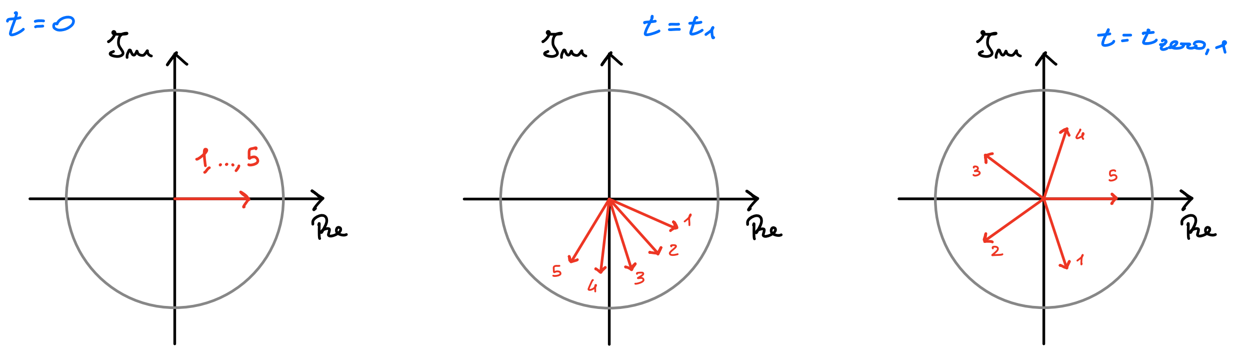
The first zero happens for $\sin\left(\frac{N\Delta\omega\tau_P}{2}\right) = 0 \Rightarrow \frac{N\Delta\omega\tau_P}{2} = \pi$

$$\Rightarrow \tau_P = \frac{2\pi}{N\Delta\omega} = \frac{1}{N\Delta\nu_c} = \frac{1}{\Delta\nu_c^*}.$$

$\Delta t_{zero} = \tau_P$, $T_R = N\tau_P$, thus, the number of zeroes within a period is $\frac{T_R}{\Delta t_{zero}} = N$.

In a phase domain view, considering for simplicity $\varphi = 0$, we have the following situation:

$E_\ell(t) = E_0 e^{j\ell\Delta\omega t}$, $\Delta\varphi = \varphi_{\ell+1} - \varphi_\ell = \Delta\omega t_1$. Considering, for example, five modes, the behaviour in time is



The fastest mode is $\ell = 5$. In $t_{zero,1} = \tau_P$, the mode 5 makes a complete turn in the complex circumference. Hence,

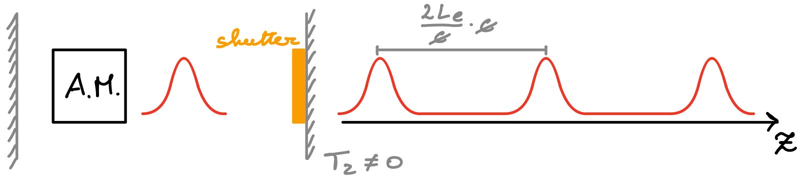
$$\varphi_5 = 5\,\Delta\omega \cdot t_{zero} = 2\pi \Rightarrow t_{zero} = \frac{2\pi}{5\cdot\Delta\omega}. \qquad \Delta\varphi = \frac{2\pi}{5}$$

For every integer multiple of τ_P we have $E_{TOT} = \sum_{\ell=1}^{5} E_\ell(t) = 0$.

When the slowest mode completes a turn, in t_m, all the modes get back in phase

$$\varphi_1 = \Delta\omega \cdot t_m = m\,2\pi \Rightarrow t_m = m\frac{2\pi}{\Delta\omega} = m\frac{1}{\Delta\nu_c}$$

Let us now study this pulse regime in the time domain. Consider the following laser structure:



The distance between two pulses is $2L_e$.

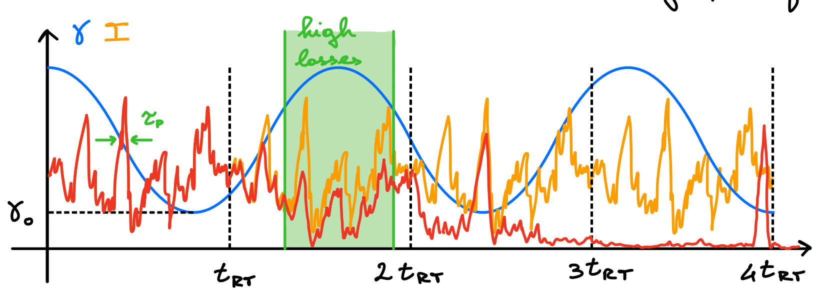
To be sure that only one pulse propagates inside the cavity we could use an optoelectronic shutter. The shutter opens up only when there's a strong pulse, in order to have gain for the pulse and high losses for other potential small pulses. This technique is known as fundamental Mode-locking.

If we put the shutter in the center of the cavity we allow two pulses to propagate. In this case we call it Harmonic Mode-locking.

How can the shutter induce the phase-locking?

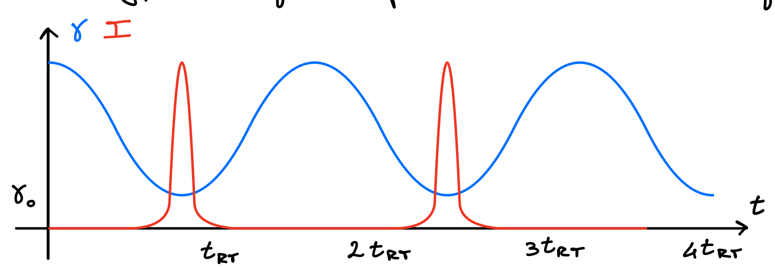
Let us assume that the modulator introduces some periodic losses $\gamma_{MOD} = \gamma_0 + \frac{\gamma'}{2}\left[1 + \sin(2\pi f_r t)\right]$, with $f_r = \frac{c}{2L_e} = \Delta\nu_c$.

Hence, the losses are modulated periodically in time (they never reach zero). When γ is high, after some round trips,



we cancel out some components of I

Eventually, the final plot becomes the following



With active Mode-locking we can produce pulses of hundreds of femtoseconds

4. RADIATION-MATTER INTERACTION

Quantum mechanics basics review

We can associate a wave function to a sub-atomical particle.
In the case of a plane wave, for example, we can write

$$\Psi(x,t) = \Psi_0 \cdot \exp\left[j\left(\frac{2\pi}{\lambda}x - 2\pi\nu t\right)\right].$$

Any wave function has to satisfy the following properties:

1) Einstein-de Broglie hypotesis: $p = \frac{h}{\lambda}$; $E_0 = h\nu$
2) The motion equation that governs this wave function must be linear.
3) Preservation of formal properties: $E_{TOT} = \frac{p^2}{2m} + U = \text{const.}$

Calling $K = \frac{2\pi}{\lambda}$; $\omega = 2\pi\nu$ and $\hbar = \frac{h}{2\pi}$ we can write the total energy and the momentum as $E_{TOT} = \hbar\omega$, $p = \hbar K$

$$\Rightarrow E_{TOT} = \frac{\hbar^2 K^2}{2m} + U$$

The wave function can be rewritten as $\Psi(x,t) = \Psi_0 e^{j(Kx-\omega t)}$

Note that:

$$\frac{\partial^2}{\partial x^2}\left[\Psi(x,t)\right] = -K^2\Psi(x,t) \Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) = \frac{\hbar^2K^2}{2m}\Psi(x,t)$$

$$\frac{\partial}{\partial t}\left[\Psi(x,t)\right] = -j\omega\Psi(x,t) \Rightarrow j\hbar\frac{\partial}{\partial t}\Psi(x,t) = \hbar\omega\Psi(x,t)$$

Hence, we can say that $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ is the kinetic energy operator, and $j\hbar\frac{\partial}{\partial t}$ is the total energy operator

For the third property ($E_{TOT} = E_{KIN} + U$) we can write the following equation, called Schrödinger equation.

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right]\Psi(x,t) = j\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

Remarks:

1) Knowing $U(x)$ in the whole space is a N.S.C. to retrieve $\Psi(x,t)$ by solving the Schrödinger equation.
2) The Schrödinger equation is complex, so the solutions $\Psi(x,t)$ are always complex.

Born's interpretation: $|\Psi(x,t)|^2 = \Psi(x,t)\cdot\Psi^*(x,t) = f(x,t)$ is the probability density to find the particle in x at a time t.

Hence, $P(x_0, x_0+\Delta x) = \int_{x_0}^{x_0+\Delta x} |\Psi(x,t)|^2 dx$, and, of course,

$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 1$ (normalization of the wave function).

Properties of the solutions of the Schrödinger equation

1) Given a potential $U(x)$ we are able to compute a discrete set of $\Psi_m(x,t)$; $m = 1, 2, 3, \ldots$ Each solution $|\Psi_m(x,t)\rangle$ can be seen as an element of a Hilbert space.

2) $|\Psi_m(x,t)\rangle = |m\rangle$ is an eigenvector of the Schrödinger equation. $|m\rangle$, $m = 1, 2, \ldots$ represents an orthonormal basis for the Hilbert space. Hence, we can say $\langle n|m\rangle = \delta_{nm}$, where

$$\langle n | m \rangle = \langle \Psi_n | \Psi_m \rangle = \int_{-\infty}^{+\infty} \Psi_n^*(x,t)\, \Psi(x,t)\, dx$$

3) Given a general wave function Ψ_g belonging to the Hilbert space, due to the Born's interpretation, we can write:

$$\| \Psi_g \| = \langle \Psi_g | \Psi_g \rangle = \int_{-\infty}^{+\infty} |\Psi(x,t)|^2 = 1$$

4) Since the equation is linear any linear combination of the eigenvectors $|\Psi_n\rangle$ is still a solution of the equation:

$$|\Psi_g(x,t)\rangle = \sum_{j=1}^{m} a_j(t)\, |\Psi_j(x,t)\rangle$$

Note: we must have $\langle \Psi_g | \Psi_g \rangle = 1 \Rightarrow \sum_{j=1}^{m} \sum_{k=1}^{m} a_j^* \langle j | \cdot a_k | k \rangle$

$$\Rightarrow \langle \Psi_g | \Psi_g \rangle = \sum_{k=1}^{m} a_k^*(t)\, a_k(t) \langle k | k \rangle = \sum_{k=1}^{m} |a_k(t)|^2 = 1.$$

Operators and expected values of physical observables

In quantum mechanics phisical observables are described by linear operators. Given an observable $\hat{A}$, we define its average value $\langle \hat{A} \rangle$ as:

$$\langle \hat{A}(t) \rangle = \int_{-\infty}^{+\infty} \hat{A} \cdot f(x,t)\, dx = \int_{-\infty}^{+\infty} \psi^*(x,t) \cdot \hat{A} \cdot \psi(x,t)\, dx = \langle \psi | \hat{A} | \psi \rangle$$

Example:

The momentum operator is defined as $\hat{p} = -j\hbar \frac{\partial}{\partial x}$, hence

$$\langle \hat{p} \rangle = \langle \psi^* | \hat{p} | \psi \rangle = -j\hbar \int_{-\infty}^{+\infty} \psi^*(x,t) \frac{\partial}{\partial x} \psi(x,t)\, dx$$

Principle of permanence of formal properties: if $\hat{a}$ is a physical observable $\langle \hat{a} \rangle$ must obey to classical laws of motion

Stationary states

Stationary states are states for which the probability distribution does not depend on time. In this case we can rewrite the wave function $\Psi(x,t)$ as: $\Psi(x,t) = \varphi(x) \cdot \eta(t)$.

If $\eta(t) = e^{j\omega t}$ we can say $|\Psi(x,t)|^2 = |\varphi(x)|^2 |e^{j\omega t}|^2 = |\varphi(x)|^2$

Putting this wave function into the Schrödinger equation:

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x} + U(x) \right] \Psi(x,t) = -j\hbar \frac{\partial}{\partial t} \Psi(x,t) \implies$$

$$\eta(t) \left[-\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x^2}(x) + U(x)\varphi(x) \right] = \varphi(x) \left[-j\hbar \frac{\partial}{\partial t} \eta(t) \right]$$

Multiplying by $\frac{1}{\varphi(x)\eta(t)}$ both the members we obtain

$$\frac{1}{\varphi(x)} \left[-\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x^2}(x) + U(x)\varphi(x) \right] = \left[-j\hbar \frac{\partial}{\partial t} \eta(t) \right] \frac{1}{\eta(t)}$$

In order for this equation to be satisfied the two terms must be equal to a constant. In particular, we can write, multiplying both terms by $\varphi(x)$:

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \varphi(x) + U(x)\varphi(x) = E\varphi(x) \implies \hat{H}|\varphi(x)\rangle = E|\varphi(x)\rangle$$

where $\hat{H} = \frac{\hbar^2}{2m} \frac{d^2}{dx^2} + U(x)$ is the Hamiltonian operator.

The last equation is called time independent Schrödinger equation and is an eigenvalues equation.

The total energy E is an eigenvalue of the equation, thus, for stationary states energy is constant.

Let us consider the temporal component of the Schrödinger eq.:

$$-j\hbar \frac{d}{dt}\eta(t) = E\eta(t) \quad \text{(first order differential equation)}$$

Multiplying both terms by $\frac{j}{\hbar\eta}$ we end up with

$$\frac{d\eta}{dt}\cdot\frac{1}{\eta} = j\frac{E}{\hbar} \Rightarrow \int\frac{d\eta}{\eta} = \int j\frac{E}{\hbar}\,dt \Rightarrow \log(\eta) = j\frac{E}{\hbar} \Rightarrow$$

$$\eta(t) = \exp\left(j\frac{E}{\hbar}t\right) = e^{j\omega t}, \quad \text{where } \omega = 2\pi\nu = \frac{E}{\hbar} \Rightarrow \nu = \frac{E}{h}$$

So, we retrived the well known relation $E = h\nu$.

Hence, the temporal component only changes the phase of the wave function, without modifying its amplitude.

Let us focus again on the time independent Schrödinger equation.

Expliciting the Hamiltonian we can write

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x)\right]\varphi(x) = E\varphi(x) \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2} + \left[U(x) - E\right]\varphi(x) = 0$$

Note that the difference between the total energy and the potential energy is the kinetic energy: $E - U(x) = U_K(x)$, so:

$$\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\varphi + U_K\varphi = 0,$$

that is analogue to the equation of an harmonic oscillator.

Time dependent perturbation theory

Let us study the interaction between matter and radiation in a semiclassical view exploiting the time dependent perturbation theory.

We saw that when light interacts with matter, there is a certain probability that a photon belonging to the EM field induces a transition, that can be an absorption or a stimulated emission. We want to define the transition rate W_{nm} between two states $|n\rangle$ and $|m\rangle$ of the unperturbed system. We know that $W_{nm} \neq 0$ only if $E_{ph} = h\nu = |E_n - E_m|$ (energy preservation principle). We will now exploit quantum mechanical arguments to find the expression of the transition rate.

In a semiclassical treatment we consider the matter as quantized and a classical electromagnetic field. We will assume the field as perfectly monochromatic, hence the interaction must have an infinite time duration. Eventually, the wavelength λ should be much higher than the size of the atom.

With these hypotesis we can write the electric field as

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\omega t)$$

We will work under the weak perturbation hypotesis: the perturbation induced by $\vec{E}$ is so small that corresponds to the energy eigenvalues of the unperturbed Hamiltonian of the system $\hat{H}$.

Let us write the Schrödinger equation with the unperturbed Hamiltonian:

$$\hat{H} \, |\psi_k(x,t)\rangle = j\hbar \frac{\partial}{\partial t} |\psi_k(x,t)\rangle$$

The perturbed system is described by the following equation:

$$\left[\hat{H} + \tilde{H}'\right] |\psi\rangle = j\hbar \frac{\partial}{\partial t} |\psi\rangle$$

where $\tilde{H}'$ is the interaction Hamiltonian. Thanks to the weak perturbation hypotesis we can say that the new solution is a linear combiation of the unperturbed states:

$$|\psi\rangle = \sum_k a_k(t) |\psi_k(x,t)\rangle,$$

where $|\psi_k(x,t)\rangle = |k\rangle e^{-j\omega_k t}$, since they're stationary states. $|k\rangle$ is the spatial component of the wave function $|k\rangle = |\psi_k(x)\rangle$ and $\omega_k = E_k/\hbar$.

Substituting $|\psi\rangle$ into the perturbed equation we obtain:

$$\left[\hat{H} + \tilde{H}'\right] \sum_k a_k(t) |\psi_k\rangle = j\hbar \frac{\partial}{\partial t} \sum_k a_k(t) |\psi_k\rangle$$

The term $[\hat{H} + \tilde{H}']$ is only space dependent, thus it can be brought into the sum:

$$\sum_k a_k(t) \hat{H} |\psi_k\rangle + \sum_k a_k(t) \tilde{H}' |\psi_k\rangle = j\hbar \sum_k \frac{da_k}{dt} |\psi_k\rangle +$$

$$+ \sum_k a_k(t) j\hbar \frac{\partial}{\partial t} |\psi_k\rangle$$

Note that $\sum_k a_k(t) \hat{H} |\psi_k\rangle = \sum_k a_k(t) j\hbar \frac{\partial}{\partial t} |\psi_k\rangle$, so the perturbed Schrödinger equation is reduced to

$$\sum_k a_k(t)\,\tilde{H}'\,|\psi_k\rangle = j\hbar \sum_k \frac{da_k}{dt}|\psi_k\rangle$$

Let us make the projection of these states on a state $|m\rangle$:

$$\langle\psi_m|\sum_k a_k(t)\,\tilde{H}'|\psi_k\rangle = \left\langle\psi_m\middle| j\hbar\sum_k \frac{da_k}{dt}\middle|\psi_k\right\rangle \Rightarrow$$

$$\sum_k a_k(t)\langle\psi_m|\tilde{H}'|\psi_k\rangle = j\hbar\sum_k \frac{da_k}{dt}\langle\psi_m|\psi_k\rangle = j\hbar\sum_k \frac{da_m}{dt}$$

The last equality is given by the fact that $\langle\psi_m|\psi_k\rangle = \delta_{mk}$.

Let us now assume that at $t=0$ the atom is prepared in the state $|n\rangle$, hence $a_k(0)=0$ except for $a_n(0)=1$.
Remember that $|a_n|^2$ represents the probability for the system to be in the state $|\psi_n\rangle$.
We will also assume that the probability for the atom to change its original state from $|n\rangle$ to $|m\rangle$, thus $a_n(t)\simeq 1$ and $a_k(t)\simeq 0$, $k\neq n$ for $t>0$. So, we can write

$$\langle\psi_m|\tilde{H}'|\psi_n\rangle = j\hbar\frac{da_m}{dt} \Rightarrow \frac{da_m}{dt} = -\frac{j}{\hbar}\langle\psi_m|\tilde{H}'|\psi_n\rangle \circledast \qquad \Rightarrow$$

$|a_m(t)|^2$ is the probability to find the atom in the state $|\psi_m\rangle$ at a time t. We are looking for an expression that gives the probability for the system to make a transition from the state $|\psi_n\rangle$ to the state $|\psi_m\rangle$, hence

$$W_{mn} = \frac{d|a_m(t)|^2}{dt}$$

To find its expression we first have to explicit the time dependance of the interaction Hamiltonian:

$$\begin{cases} \tilde{H}' = H_0' \cdot \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) = H_0' \cos(\omega t). \\ |\psi_n\rangle = |n\rangle e^{-j\omega_n t}; \quad \langle\psi_n| = \langle n| e^{j\omega_n t}, \quad \text{where } \omega_n = \frac{E_n}{\hbar} \\ |\psi_m\rangle = |m\rangle e^{-j\omega_m t}; \quad \langle\psi_m| = \langle m| e^{j\omega_m t}, \quad \text{where } \omega_m = \frac{E_m}{\hbar} \end{cases}$$

Thus, we can rewrite the equation ⊛ as:

$$\frac{da_m}{dt} = -\frac{j}{2\hbar}\langle m|H_0'|n\rangle \cdot \left[e^{j(\omega_0+\omega)t} + e^{-j(\omega-\omega_0)t}\right],$$

where $\omega_0 = \omega_m - \omega_n = \frac{E_m - E_n}{\hbar} = \omega_{mn}$ is the transition frequency between n and m.

At resonance between ω and ω_0 we have $\omega \simeq \omega_0$ ($\omega_0 + \omega$ will be of the order of 10^{14} Hz, while $\omega - \omega_0$ will be of the order of 10^2 Hz). Let us now integrate the former equation:

$$\int_0^t da_m = -\frac{j}{2\hbar}\int_0^t \langle m|H_0'|n\rangle \cdot \left[e^{j(\omega_0+\omega)\tau} + e^{-j(\omega-\omega_0)\tau}\right] d\tau \Rightarrow$$

$$a_m(t) - \underbrace{a_m(0)}_{=0} = -\frac{j\langle m|H_0'|n\rangle}{2\hbar}\left[\int_0^t e^{j(\omega_0+\omega)\tau} d\tau + \int_0^t e^{-j(\omega-\omega_0)\tau} d\tau\right]$$

We can call $\langle m|H_0'|n\rangle = H'_{mn}$, that is the expected value of the interaction energy during the transition.

$$a_m(t) = -\frac{j}{2\hbar} H'_{mn}\left[\frac{e^{j(\omega_0+\omega)t} - 1}{j(\omega_0+\omega)} - \frac{e^{-j(\omega-\omega_0)t} - 1}{j(\omega-\omega_0)}\right].$$

The first addend is negligible w.r.t. the second (12 orders of magnitude of difference). Hence,

$$a_m(t) = \frac{j}{2\hbar} H'_{mn} \frac{e^{-j\Delta\omega t} - 1}{j\Delta\omega}.$$ Multiplying by $\frac{2t}{2t} e^{j\frac{\Delta\omega}{2}t} e^{-j\frac{\Delta\omega}{2}t}$:

$$a_m(t) = -\frac{j}{2\hbar} H'_{mn} e^{-j\frac{\Delta\omega}{2}t} \cdot t \cdot \frac{e^{-j\frac{\Delta\omega}{2}t} - e^{j\frac{\Delta\omega}{2}t}}{2j\frac{\Delta\omega t}{2}} \Rightarrow$$

$$a_m(t) = -\frac{j}{2\hbar} H'_{mn} e^{j\frac{\Delta\omega}{2}t}\, t\, \mathrm{sinc}\left(\frac{\Delta\omega t}{2\pi}\right).$$ $$\left[\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}\right]$$

$$|a_m(t)|^2 = |H_{mn}|^2 \cdot \frac{t^2}{4\hbar^2}\, \mathrm{sinc}^2\left(\frac{\Delta\omega t}{2\pi}\right)$$

The interaction takes place for $t \to \infty$, and for $t \to \infty$ the sinc in frequency becomes a delta function:

$$\lim_{t\to\infty} \left\{ t\, \mathrm{sinc}\left(\frac{\Delta\omega t}{2\pi}\right)\right\} = 2\pi \cdot \delta(\Delta\omega).$$ Thus,

$$|a_m(t)|^2 = \frac{\pi}{2\hbar^2} t\, |H'_{mn}|^2 \delta(\Delta\omega) \Rightarrow W_{mn} = \frac{d|a_m(t)|^2}{dt} = \frac{\pi}{2\hbar^2} |H'_{mn}|^2 \delta(\Delta\omega)$$

The former expression tells us that the transition can happen only if $\Delta\omega = 0$.

Let us now find an expression for the interaction energy H'_{mn}:

$H'_{mn} = \langle m | H'_0 | n \rangle = \vec{\mu}_{mn} \cdot \vec{E}_0$, where $\vec{\mu}_{mn}$ is the classical dipole moment of the atom during the transition $|n\rangle \to |m\rangle$.

$\vec{\mu}_{mn} = \langle m | q \cdot \vec{x} | n \rangle$, $\vec{x}$ is the displacement between the electron in $\vec{x}$ and the nucleus in $\vec{x} = 0$. Hence,

$$|H'_{mn}|^2 = |\vec{\mu}_{mn}|^2 \cdot |\vec{E}_0|^2 \cos^2(\vartheta)$$

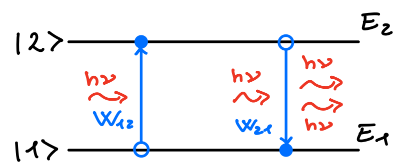
$$\langle |H'_{mn}|^2 \rangle_{\Omega=4\pi} = |\vec{\mu}_{mn}|^2 \cdot |\vec{E}_0|^2 \langle \cos^2(\vartheta) \rangle_{\Omega=4\pi} = \frac{1}{3} |\vec{\mu}_{mn}|^2 \cdot |\vec{E}_0|^2$$

Substituting this result into the transition rate we obtain:

$$W_{mn} = \frac{\pi}{6\hbar^2} |\vec{\mu}_{mn}|^2 E_0^2 \, \delta(\Delta\omega) = \frac{\pi^2}{3h^2} |\vec{\mu}_{mn}|^2 E_0^2 \, \delta(\Delta\nu),$$

since $\delta(\Delta\omega) = 2\pi \delta(\Delta\nu)$.

Remarks on the Fermi's golden rule



We saw that given an atom in a low energy state $|1\rangle$, the transition rate given by the interaction between a photon $h\nu = E_2 - E_1$ and the atom is given by

$$W_{12} = \frac{\pi^2}{3h^2} |\vec{\mu}_{12}|^2 E_0^2 \, \delta(\Delta\nu).$$ This is called Fermi's golden rule.

Some remarks on the rule:

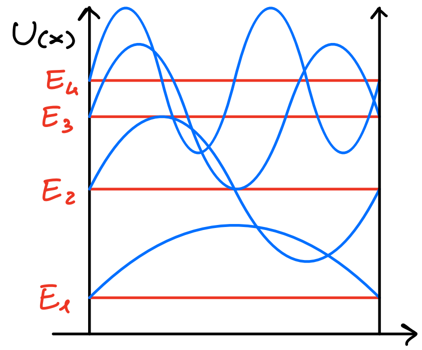
1) The stimulated emission rate is $W_{21} = \frac{\pi^2}{3h^2} |\vec{\mu}_{21}|^2 E_0^2 \, \delta(\Delta\nu)$.

$$\vec{\mu}_{21} = \langle 2 | q \cdot \vec{x} | 1 \rangle = \int \varphi_2^*(\vec{x}) \cdot q\vec{x}\, \varphi_1(\vec{x})\, d\vec{x} = \vec{\mu}_{12}^*, \quad \text{Hence,}$$

$$|\vec{\mu}_{21}|^2 = |\vec{\mu}_{12}^*|^2 = |\vec{\mu}_{12}|^2 \Rightarrow W_{21} = W_{12}$$

2) If $\vec{\mu}_{mn} = 0$ is forbidden by the electrical dipole moment.
All the stationary eigenstates $|k\rangle$ of the isolated atom must have a well defined parity. So, given a wave function $\varphi(x)$ we must have $\varphi_k(-x) = \varphi_k(x)$ or $\varphi_k(-x) = -\varphi_k(x)$ (even or odd eigenfunction).

In a square well we have an alternation between even and odd parity, as schematized in the figure aside.



The dipole moment is given by

$$\vec{\mu}_{mn} = \langle m | q\vec{x} | n \rangle = \int \varphi_m^*(\vec{x})\, q\vec{x}\, \varphi_n(\vec{x})\, d\vec{x}$$

Hence, $\vec{\mu}_{mn} = 0$ if φ_m and φ_n have the same parity; we can have transitions only between states with different parity.

3) We can find some useful equivalent expressions for W_{mn}. E.g., in a medium with refractive index n the intensity of an electromagnetic wave is given by

$$I = \frac{1}{2}\varepsilon_0 c_0 n E_0^2 \Rightarrow W_{12} = \frac{2\pi^2 |\mu_{12}|^2}{3\varepsilon_0 c_0 n h^2} \cdot I \cdot \delta(\Delta\nu).$$

We said, in chapter 3, that $W_{12} = \sigma_{12} \cdot \Phi$ and $I = \Phi h\nu$

$$W_{12} = \frac{2\pi^2 \nu}{3\varepsilon_0 c_0 n h} |\mu_{12}|^2 \Phi\, \delta(\nu - \nu_0)$$

Calling ρ the EM density of energy, $I = \rho \frac{c_0}{n}$, hence

$$W_{12} = \frac{2\pi^2}{3\varepsilon_0 n^2 h^2} |\mu_{12}|^2 \rho\, \delta(\nu - \nu_0) = B_{12}\, \rho(\nu), \quad \text{where}$$

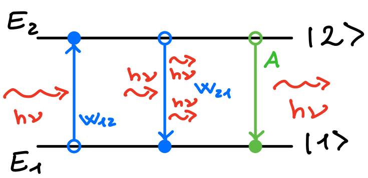
$$B_{mn} = \frac{2\pi^2}{3\varepsilon_0 n^2 h^2} |\mu_{mn}|^2 \delta(\nu - \nu_0)$$ is called Einstein coefficient for the transition $|n\rangle \to |m\rangle$

Einstein coefficient for the spontaneous emission

In the former paragraph we defined the Einstein coefficient for the absorption B_{mn}. Let us now find the coefficient for the spontaneous emission A.

In chapter 3 we wrote the rate equation for spontaneous emission:

$$\frac{dN_2}{dt} = A N_2 = \frac{N_2}{\tau_{sp}}, \quad \text{where } A = \frac{1}{\tau_{sp}} \text{ is the Einstein coefficient.}$$



Let us use a thermodynamic approach in which we consider two level atoms in thermodynamic equilibrium in a tank with a fixed temperature T. The system exchanges energy by emitting and absorbing photons.

At thermodynamic equilibrium we can say that

$$\frac{dN_1}{dt} = \left(\frac{dN_2}{dt}\right)_{TOT} \Rightarrow W_{12} N_1 = (W_{21} + A) N_2 \Rightarrow$$

$$B\rho(\nu) N_1 = B\rho(\nu) N_2 + A N_2 \Rightarrow B\rho(\nu)(N_1 - N_2) = A N_2 \Rightarrow$$

$$\rho(\nu) = \frac{A N_2}{B(N_1 - N_2)} = \frac{A}{B} \cdot \frac{1}{N_1/N_2 - 1}.$$

According to the Boltzmann statistics $\frac{N_1}{N_2} = e^{\frac{E_2 - E_1}{KT}} = e^{\frac{h\nu}{KT}}$

Hence, $\rho(\nu) = \frac{A}{B} \cdot \frac{1}{e^{\frac{h\nu}{KT}} - 1}$, that is the well known energy density expression for the black-body radiation:

$$\rho(\nu) = h\nu \frac{8\pi\nu^2}{c_m^2} \cdot \frac{1}{e^{h\nu/KT} - 1}$$

Thus, we can say that $\frac{A}{B} \cdot \frac{1}{e^{h\nu/kT}-1} = \frac{8\pi\nu^3 n^3 h}{c_0^3} \frac{1}{e^{h\nu/kT}-1}$

$\Rightarrow A = B \cdot \frac{8\pi\nu^3 n^3 h}{c_0^3}$. Substituting the expression of B we have

$$A(\nu) = \frac{16\pi^3 n\nu^3 |\vec{\mu}_{21}|}{3\varepsilon_0 c_0^3 h} \delta(\nu-\nu_0)$$

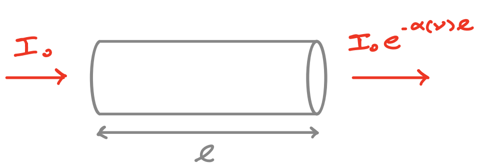
Integrating over the all the frequencies we obtain

$$A_{mn} = \int_{-\infty}^{+\infty} \frac{16\pi^3 n\nu^3 |\vec{\mu}_{21}|}{3\varepsilon_0 c_0^3 h} \delta(\nu-\nu_0)\, d\nu = \frac{16\pi^3 n\nu^3 |\vec{\mu}_{21}|}{3\varepsilon_0 c_0^3 h}$$

$$\tau_{sp} = \frac{1}{A_{mn}} = \frac{3\varepsilon_0 c_0 h}{16\pi^3 n\nu^3 |\vec{\mu}_{21}|^2}$$

Sources of homogeneous broadening of the σ_{12}

Let us consider a gas cell crossed by a beam of intensity I_0.



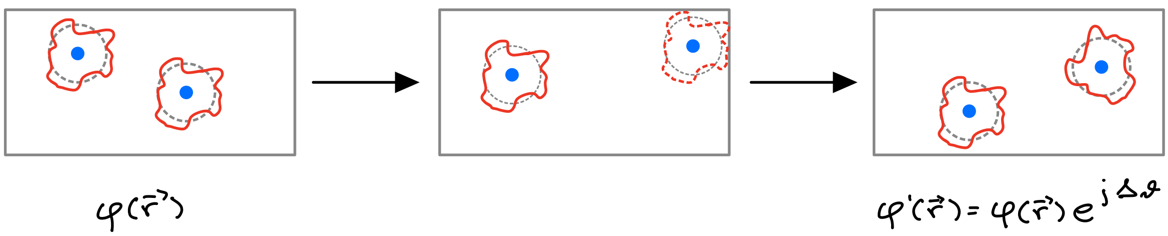
$\alpha(\nu) = \sigma_{12}(\nu) N_1 \Rightarrow I_{OUT} = I_0 e^{-\sigma_{12}(\nu) N_1 \ell}$

$\Rightarrow \sigma_{12}(\nu) = -\frac{1}{N_1 \ell} \ln\left(\frac{I_{OUT}}{I_0}\right)$

In a real experiment σ_{12} is not a δ but there is a spectral broadening, as we saw in chapter 3. We divided these broadenings into two kinds: homogeneous and inhomogeneous. In the first case σ_{12} has a Lorentian-like shape, in the second case we end up with an overall $\sigma^*_{12,inhomog}(\nu)$ that has a gaussian shape. Let us first study the causes of homogeneous broadening.

1) Collisional broadening.

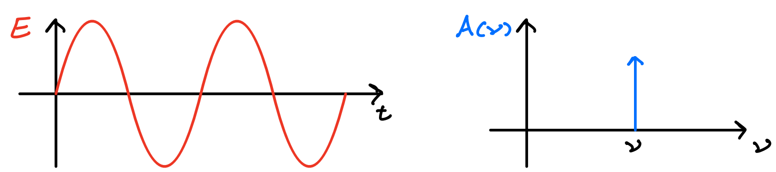
Let us consider an atomic sample into a gas cell. Due to thermal energy, the molecules are subject to a translational motion. A molecule can hit a wall of the container or collide to another molecule. After a collision the wave function is



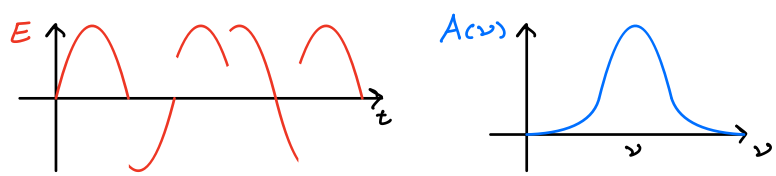
becomes non-stationary (temporarly disrupted) and after the collision the wave function is re-estabilished with a random phase jump w.r.t. the former one.

Let us consider these two reference frames:

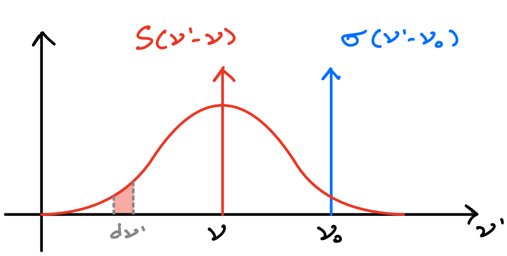
a) Fixed with the cell



b) Fixed with the atom



We call $\langle \tau \rangle = \tau_c$ the stochastic average of the time between two collisions.



We call $S(\nu)$ the intensity spectrum.

$$dI = S(\nu'-\nu)\, d\nu' \Rightarrow$$

$$I_0 = \int_{-\infty}^{+\infty} S(\nu'-\nu)\, d\nu' \Rightarrow$$

$$\Rightarrow \int_{-\infty}^{+\infty} g(\nu'-\nu)\, d\nu' = 1, \quad \text{where} \quad g(\nu'-\nu) = \frac{S(\nu'-\nu)}{I_0}.$$

Hence, since $W_{12} = c\, I\, \delta(\nu'-\nu_0)$, with $c = \dfrac{2\pi^2 |\vec{\mu}_{12}|^2}{3\varepsilon_0 c_0 n h^2}$, we can

write $dW_{12} = c\, dI\, \delta(\nu'-\nu_0) = c\, I_0\, \bar{g}(\nu'-\nu)\, \delta(\nu'-\nu_0)\, d\nu' \Rightarrow$

$$W_{12} = \int_{-\infty}^{+\infty} c\, I_0\, \bar{g}(\nu'-\nu)\, \delta(\nu'-\nu_0)\, d\nu' = c\, I_0\, \bar{g}(\nu_0-\nu), \quad \text{with } \bar{g}(\nu) = g(-\nu)$$

$$\Rightarrow W_{12} = \frac{2\pi\, |\vec{\mu}_{12}|}{3\varepsilon_0 c_0 n h^2}\, I_0\, g(\nu-\nu_0)$$

This is the homogeneous broadened transition rate.

$g(\nu-\nu_0)$ is the normalized intensity spectrum. We have to prove that this spectrum is Lorentian.

$$g(\nu-\nu_0) = \frac{S(\nu-\nu_0)}{I_0}, \quad \text{but} \quad S(\nu) = \mathcal{F}\{\Gamma(\tau)\}$$

Random phase jumps take place with a probability density as a function of τ (time between two collisions) given by

$$p(\tau) = \frac{1}{\tau_c} e^{-\frac{\tau}{\tau_c}}, \quad \text{where} \quad \tau_c = \langle \tau \rangle = \frac{1}{\tau}\int_0^{\tau} p(t)\, dt.$$

In this case $\Gamma(\tau) = E_0^2 e^{j\omega\tau} e^{\tau/\tau_c}$.

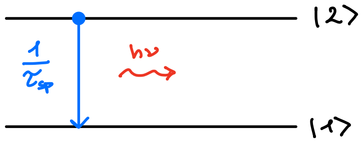
$$S(\nu'-\nu) = \mathcal{F}\{\Gamma(\tau)\} = I_0 \frac{2\tau_c}{4\pi^2 \tau_c^2 (\nu'-\nu)^2 + 1} \Rightarrow$$

$$g(\nu-\nu_0) = \frac{2\tau_c}{4\pi^2\tau_c^2(\nu-\nu_0)^2 + 1},$$ that is a Lorentian shape

with $\Delta\nu_0 = \dfrac{1}{\pi\tau_c}$

2) Spontaneous broadening

Let us consider a material with two levels.



An atom in state $|2\rangle$ can spontaneously decay to state $|1\rangle$ with the equation

$$\left.\frac{dN_2}{dt}\right|_{sp} = AN_2 = \frac{N_2}{\tau_{sp}} \Rightarrow \frac{dN_2}{N_2} = \frac{dt}{\tau_{sp}} \Rightarrow N_2(t) = N_2(0)\, e^{-\frac{t}{\tau_{sp}}}$$

$$I(t) \propto \frac{dN_2}{dt} = \frac{N_2(t)}{\tau_{sp}} = \frac{N_2(0)}{\tau_{sp}}\, e^{-t/\tau_{sp}} \Rightarrow$$

$$E(t) = E_0\, e^{-j\omega t}\, e^{-\frac{t}{2\tau_{sp}}}, \quad \text{since } |E(t)| \propto \sqrt{I(t)}$$

Hence, the amplitude spectrum $A(\nu) = \mathcal{F}\{E(t)\}$ will be given by

$$A(\nu) = \mathcal{F}\{E(t)\} \propto \frac{1}{4\pi^2 \tau_{sp}(\nu-\nu_0)^2 + 1}, \quad \text{where } \nu_0 = \frac{E_2 - E_1}{h}$$

In this case $\Delta\nu_{sp} = \frac{1}{2\pi\tau_{sp}} \Rightarrow \Rightarrow \Delta\nu_{sp}\cdot\tau_{sp} = \frac{1}{2\pi}$

This last relation represents the indetermination on the photon frequency. Calling $\Delta E = h\Delta\nu_{sp}$ the indetermination on the photon energy:

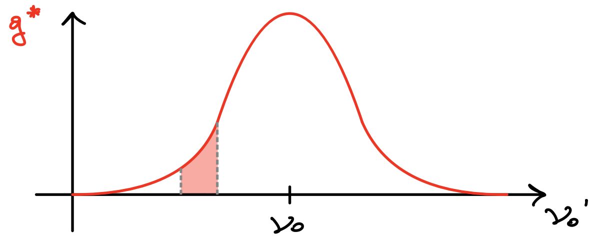
$\Delta\nu_{sp}\,\tau_{sp} = \frac{1}{2\pi} \Rightarrow h\Delta\nu_{sp}\,\tau_{sp} = \hbar \Rightarrow \Delta E\cdot\Delta t = \hbar$, that is the indetermination principle in the energy form

Sources of inhomogeneous broadening

1) Stark effect (typical of solid state lasers).

It is due to local field variations on the emitting/absorbing atoms. Stark effects are high for transition metals and is low for rare earths.

Crystals are subject to low Stark effects (it only happens for the defects), while is much higher for amorphous environments.



$$g^*(\nu_0'-\nu_0)\,d\nu_0' = \frac{dN(\nu_0')}{N_{TOT}}$$

$g^*(\nu'-\nu_0)\,d\nu_0'$ is the fraction of atoms with resonance frequency between ν_0 and $\nu_0 + d\nu_0'$

$$\int_0^{+\infty} g^*(\nu_0'-\nu_0)\,d\nu_0' = \frac{N_{TOT}}{N_{TOT}} = 1$$

g^* has a gaussian shape due to the law of large numbers

$$dW_{12} = c I_0\, \delta(\nu-\nu_0')\, g^*(\nu_0'-\nu_0)\, d\nu_0' \Rightarrow$$

$$W_{12} = \int_{-\infty}^{+\infty} c I_0\, \delta(\nu-\nu_0')\, g^*(\nu_0'-\nu_0)\, d\nu_0' = c I_0\, g^*(\nu-\nu_0) \Rightarrow$$

$$W_{12} = \frac{2\pi^2 |\vec{\mu}_{12}|^2}{3\varepsilon_0 c_0 m h^2} I_0\, g^*(\nu-\nu_0).$$

ν is the frequency of incoming photon and ν_0 is tre resonance frequency. g^* represents the inhomogeneous gaussian line shape.

2) Doppler broadening (typical of gas laser).

Calling $\vec{v}_{em} = \frac{c_0}{m}\vec{v}_z$ the velocity of an electromagnetic wave and $\vec{v}_z$ the velocity of an atom along the z-axis, the relative frequency will be given by

$$\nu_{rel} = \nu\left[1 - \frac{v_z}{c_m}\right] = \nu_0 = \nu_0'\left[1 - \frac{v_z}{c_m}\right] \Rightarrow \nu_0' = \frac{1}{\left[1-{v_z}/{c_m}\right]}$$

ν_0' is the frequency of the photon that induced the transition.

The probability for an atom to have a velocity v_z is given by the Maxwell's distribution of velocity at temperature T:

$$p(v_z) = \sqrt{\frac{M}{2\pi KT}}\ \exp\left[-\frac{M v_z^2}{2KT}\right] \Longrightarrow$$

$$\nu_0' = \frac{\nu_0}{1-v_z/c_m}\cdot\frac{1+v_z/c_m}{1+v_z/c_m} = \frac{\nu_0}{1-v_z^2/c_m^2}\left(1+\frac{v_z}{c_m}\right) \Longrightarrow v_z = c_m\left(\frac{\nu_0'}{\nu_0}-1\right) \Longrightarrow$$

$$dv_z = \frac{c_m}{\nu_0}\,d\nu_0'.$$

$$g^*(\nu_0'-\nu_0)\,d\nu_0' = p(v_z)\,dv_z = \sqrt{\frac{M}{2\pi KT}}\ \exp\left[-\frac{M}{2KT}c_m\left(\frac{\nu_0'}{\nu_0}-1\right)^2\right]\cdot\frac{c_m}{\nu_0}\,d\nu_0'$$

$$\Rightarrow g^*(\nu_0'-\nu_0) = \frac{1}{\nu_0}\sqrt{\frac{Mc_m^2}{2\pi KT}}\ \exp\left\{\frac{Mc_m^2}{2KT}\cdot\frac{(\nu_0'-\nu_0)^2}{\nu_0^2}\right\}$$

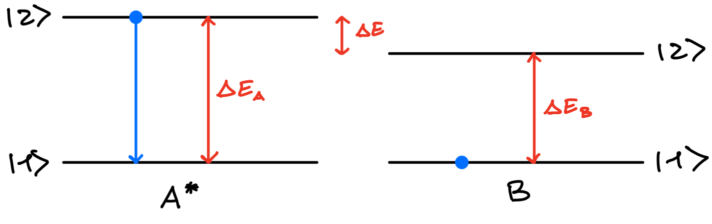
So, g^* has a Gaussian shape with $\Delta\nu_0^* = 2\nu_0\sqrt{\dfrac{2KT\ln 2}{Mc_m^2}}$

Non-radiative decay

The main causes of non-radiative decay are the following.

1) Collisional deactivation (mainly for gas form).

Consider two species A and B with different energy levels and assume that A is in the excited state (A^*) and B is in the ground state.



If the two species collide it can happen a super-elastic collision, i.e.:

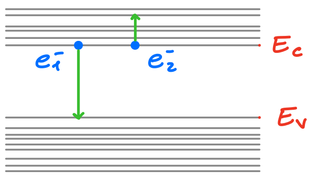
$$A^* + B = A + B + \Delta E_{KIN}.$$

In a super-elastic collision there is an increasing of the kinetic energy of the system. This process happens if

$\Delta E = \Delta E_A - \Delta E_B \gg KT$ and $m_B \ll m_A$

2) Auger ricombination (mainly in semiconductor)

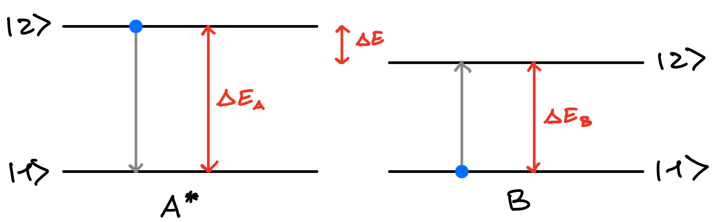
Consider the valence and the conduction band of a material.



$1^* + 2^* \rightarrow 1 + 2^{**}$

If two electrons are at the bottom of the conduction band and collide, e_1^- can get de-excited and e_2^- more excited.

3) Quasi-resonant energy transfer (both in gas phase and solid state lasers).



It is a situation similar to collisional deactivation, but in this case $\Delta E \simeq KT$

$A^* + B = A + B^* + \Delta E_K$.

In this case ΔE_K is very small and may appear as vibrational energy.

There are two possible energy transfers between molecules or excitons: short range (or Dexter) and long range (or Forster) energy transfer.

In the first case the two species must be so close that the wave functions of the valence electrons overlap. This means that the tails of an electron's wave function overlap with the tail of the other one.

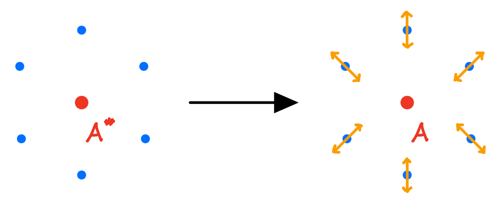
In the Forster transfer, A^* is excited and can decay radiatively to the ground state and B is in the ground state and can go stimulated absorption. It happens when the emission spectrum of A^* well overlaps with the absorption spectrum of B.

The Dexter transfer happens if the two species distance is in the order of the Ångström, while the Forster happens if the distance is up to 10 nm.

4) Thermal deactivation (mainly in solid state).

Let us consider an active ion A^* surrounded by the ions of the crystal lattice. The excitation energy can be transferred to the host matrix in form of phonons. Calling Ω the resonance frequency of the phonons, we can write:

$A^* \rightarrow A + \sum h\Omega$, where $h\Omega$ is the phonon's energy.



The ion gets de-excited ($A^* \rightarrow A$) and the ions around start vibrating.

This phenomenon is typical of the transition metal ions.

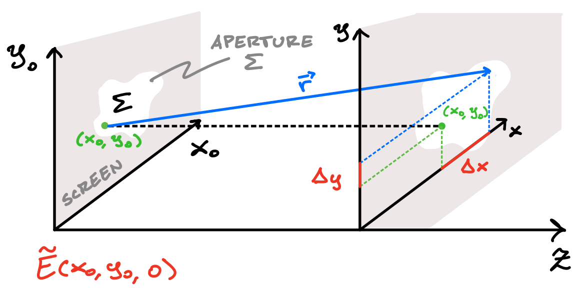
5. SCALAR THEORY OF DIFFRACTIVE LIGHT PROPAGATION

Elements of diffraction theory

The first formal definition of diffraction was given by A. Sommerfeld: "Diffraction is any deviation of light rays from the rectilinear path that can not be described as reflection or refraction".

We can typically see this behaviour when light crosses small apertures (comparable with the wavelength). In this case we can see that the projected image is larger than the hole and we observe diffraction pattern (rings).

Let us try to mathematically describe the problem.



We will make some hypotesis:

1) The $\vec{E}$ field is linearly polarized (it can be considered a scalar).

2) The field is perfectly monochromatic: $\tilde{E}(\vec{r}, t) = \tilde{E}(\vec{r})\, e^{-j\omega t}$

Let us put this field into the wave equation:

$$\nabla^2 \tilde{E}(\vec{r}, t) - \frac{1}{c_m^2} \frac{\partial^2}{\partial t^2} \tilde{E}(\vec{r}, t) = 0 \quad \Rightarrow$$

$$\nabla^2 \tilde{E}(\vec{r}) + \left(\frac{\omega}{c_m}\right)^2 \tilde{E}(\vec{r}) = 0 \quad \Rightarrow \quad \left[\nabla^2 + K\right] \tilde{E}(\vec{r}) = 0,$$

where $K \equiv \frac{\omega^2}{c_m^2}$. This relation is called Helmoltz equation.

Integrating the equation we obtain

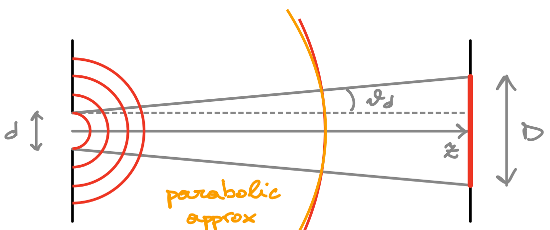
$$\tilde{E}(x,y,z) = \frac{1}{j\lambda} \iint_{\Sigma} \tilde{E}(x_0, y_0, 0) \cdot \frac{e^{jkr}}{r} \cos(\hat{u}_z \cdot \vec{r})\, dx_0\, dy_0$$

where $\vec{r} = (x-x_0)\hat{u}_x + (y-y_0)\hat{u}_y + z\,\hat{u}_z$ and $r \equiv |\vec{r}|$

The former result is called Sommerfeld-Kirchhoff-Rayleigh integral, and represents mathematically the Huygens interpretation of diffraction.

To analytically solve this problem we should make a couple of approximations:

1) Paraxial approximation (or Fresnel, or near field).



If D is small and z is large enough, we can assume $\vartheta_d \simeq \tan(\vartheta_d) \simeq \frac{D}{z}$

In this case, $\cos(\hat{u}_z \cdot \vec{r}) \simeq 1$. Furthermore, if we are far from the source we can approximate the spherical phase-front with a parabolic phase-front (second order Taylor polynomial).

$$r = |\vec{r}| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2} = z\sqrt{1 + \frac{(x-x_0)^2}{z^2} + \frac{(y-y_0)^2}{y^2}} \Rightarrow$$

$$r \simeq z\left[1 + \frac{(x-x_0)^2}{2z^2} + \frac{(y-y_0)^2}{2y^2}\right]$$

This approximation holds if $(x-x_0) \ll z$ and $(y-y_0) \ll z$, hence we must have $d \ll z$ (of an order of magnitude is enough).

Rewriting the integral with the aforementioned approximations we obtain:

$$\tilde{E}(x,y,z) = \frac{e^{jkz}}{j\lambda z}\iint_{\mathbb{R}^2} \tilde{E}(x_0,y_0,0)\exp\left[\frac{jk}{2z}\left((x-x_0)^2+(y-y_0)^2\right)\right]dx_0\,dy_0 .$$

This is the Sommerfeld integral in paraxial approximation.

2) Fraunhofer approximation (or far field).

This approximation holds if $z \gg \frac{\Sigma}{\lambda}$ (hence, if $d = 1\,mm$ we should have $z \gg 10\,m$).

Let us start with the integral in Fresnel approximation:

$$\vec{E}(\vec{r}) = \frac{e^{jkz}}{j\lambda z}\iint_{\mathbb{R}^2} \tilde{E}(x_0,y_0,0)\cdot\exp\left[j\frac{k}{2z}\left(x^2+y^2+x_0^2+y_0^2-2x_0x-2y_0y\right)\right]dx_0\,dy_0$$

$$= \frac{e^{jkz}}{j\lambda z}\cdot e^{j\frac{k}{2z}(x^2+y^2)}\iint_{\mathbb{R}^2} \tilde{E}(x_0,y_0,0)\, e^{j\frac{\pi}{\lambda z}(x_0^2+y_0^2)}\, e^{-j\frac{2\pi}{\lambda z}(xx_0+yy_0)}\,dx_0\,dy_0$$

We can say that $\pi(x_0^2+y_0^2) = \pi r_0^2 < \Sigma \Rightarrow \frac{\pi r_0^2}{\lambda z} < \frac{\Sigma}{\lambda z}$.

Since we assumed $z \gg \frac{\Sigma}{\lambda} \Rightarrow \frac{\Sigma}{\lambda z} \ll 1 \Rightarrow \frac{\pi(x_0^2+y_0^2)}{\lambda z} \ll 1$

Hence, for the Fraunhofer approximation, $e^{j\frac{\pi}{2z}(x_0^2+y_0^2)} \to 1$

$$\Rightarrow \tilde{E}(\vec{r}) = \underbrace{\frac{e^{jkz}}{j\lambda z}\cdot e^{j\frac{k}{2z}(x^2+y^2)}}_{\text{phase term}}\iint_{\mathbb{R}^2} \tilde{E}(x_0,y_0,0)\, e^{-j2\pi\left(\frac{x}{\lambda z}x_0+\frac{y}{\lambda z}y_0\right)}\,dx_0\,dy_0$$

Calling f_x and f_y the spatial frequencies of the field $\tilde{E}(x,y,z)$ defined by $f_x = \frac{x}{\lambda z}$ and $f_y = \frac{y}{\lambda z}$ we can rewrite:

$$\tilde{E}(f_x,f_y,z) = \frac{e^{jkz}}{j\lambda z}\cdot e^{j\frac{k}{2z}(x^2+y^2)}\iint_{\mathbb{R}^2} \tilde{E}(x_0,y_0,0)\, e^{-j2\pi\left(f_x x_0+f_y y_0\right)}\,dx_0\,dy_0$$

The last integral is a 2D-Fourier transform, so we can say that

$$\tilde{E}(f_x, f_y, z) = (\text{phase term}) \cdot \mathcal{F}_{2D}\left\{\tilde{E}(x_0, y_0, z)\right\}$$

The intensity of the field is given by:

$$I = \frac{1}{2}\varepsilon_0 c_0 n \left|\tilde{E}(x, y, z)\right|^2 = \frac{1}{2}\varepsilon_0 c_0 n \left|\mathcal{F}_{2D}\left\{\tilde{E}(x_0, y_0, z)\right\}\right|$$

Remarks about the 2D Fourier transform

In the 1D Fourier transform we usually work with a temporal variable, making an infinite sum of weighted temporal harmonics:

$$g(t) = \int_{-\infty}^{+\infty} \tilde{G}(f)\, e^{j2\pi f t}\, df, \quad \text{where } \tilde{G}(f) = \mathcal{F}\{g(t)\}$$

In the 2D Fourier transform, we can think to a function $g(x, y)$ as an infinite sum of weighted spatial harmonics:

$$g(x, y) = \iint_{\mathbb{R}^2} \tilde{G}(f_x, f_y)\, e^{j2\pi(f_x x + f_y y)}\, df_x\, df_y, \quad \text{where}$$

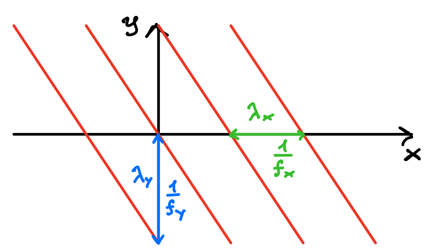
$$G(f_x, f_y) = \mathcal{F}_{2D}\{g(x, y)\} = \int_{-\infty}^{+\infty} g(x, y)\, e^{-j2\pi(f_x x + f_y y)}\, dx\, dy$$

Let us understand the behaviour in space of spatial harmonics.

$\tilde{h}(x, y) = e^{j2\pi(f_x x + f_y y)}$. We have phase maxima where

$$2\pi(f_x x + f_y y) = m \cdot 2\pi \quad \Rightarrow \quad y = -\frac{f_x}{f_y} x + \frac{m}{f_y}$$

Thus, we can represent the phase maxima as in the plot aside.



Let us consider a wave with propagation vector $\vec{k}$:

$$\vec{k} = (k_x, k_y, k_z) = |\vec{k}|\cos(\vartheta_x) + |\vec{k}|\cos(\vartheta_y) + |\vec{k}|\cos(\vartheta_z) =$$

$= \alpha|\vec{k}| + \beta|\vec{k}| + \gamma|\vec{k}|$

$$\tilde{h}(x, y, z) = e^{j\vec{k}\cdot\vec{r}} = e^{j(k_x x + k_y y + k_z z)} = e^{j2\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y + \frac{\gamma}{\lambda}z\right)}$$

It can be easily proved that $\alpha^2 + \beta^2 + \gamma^2 \Rightarrow \gamma = \sqrt{1-(\alpha^2+\beta^2)}$

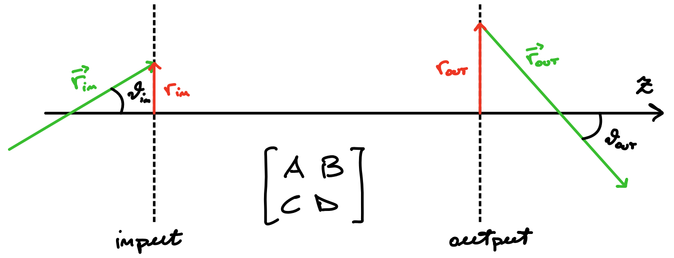
With the paraxial approximation, $\gamma = \cos(\vartheta_z) = 1 - \frac{\alpha^2+\beta^2}{2}$

Projecting the wave on the plane $z = 0$ we obtain

$$\tilde{h}(x_0, y_0, 0) = e^{j2\pi\left(\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y\right)} = e^{j2\pi(f_x x + f_y y)},$$ where $f_x = \frac{\alpha}{\lambda}$; $f_y = \frac{\beta}{\lambda}$

Matrix approach to geometrical optics

Let us consider an optical system with an input and an output plane.



If we are in paraxial approximation, the system is linear and it can be described by a matrix called ABCD matrix

$$\begin{bmatrix} r_{out} \\ \vartheta_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_{in} \\ \vartheta_{in} \end{bmatrix}$$

In paraxial approximation $\vartheta = \frac{dr}{dz} = r'(z)$: $\vartheta_{in} = r'_{in}$ and $\vartheta_{out} = r'_{out}$

This happens since $\vartheta \simeq \tan\vartheta = \frac{\Delta r}{\Delta z}$

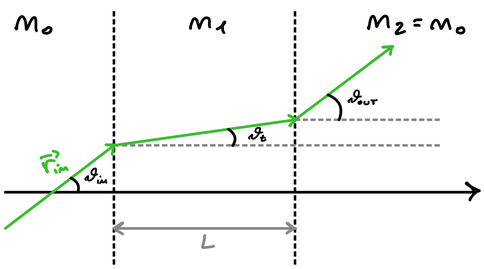
Hence, the relation above can be rewritten as:

$$\begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix}$$

We will follow the following convention: beams that propagate upward have positive displacement and the angle is positive.

Examples.

Let us consider the ABCD matrix of the free space propagation.



$$\sin\vartheta_t = \frac{n_0}{n_1} \cdot \sin\vartheta_{in} \simeq \frac{n_0}{n_1} \tan\vartheta_{in}$$

$$\Rightarrow \sin\vartheta_t \simeq \frac{n_0}{n_1} r'_{in}$$

$$\Delta r = L \cdot \tan\vartheta_t \simeq L \sin\vartheta_t = \frac{n_0}{n_1} L r'_{in}$$

$$r_{out} = r_{in} + \Delta r = r_{in} + \frac{n_0}{n_1} L r'_{in}$$

$$r'_{out} = \sin\vartheta_{out} = \frac{n_1}{n_2} \sin\vartheta_t = \frac{n_1}{n_2} \cdot \frac{n_0}{n_1} r'_{in} \Rightarrow r'_{out} = \frac{n_0}{n_2} r'_{in}$$

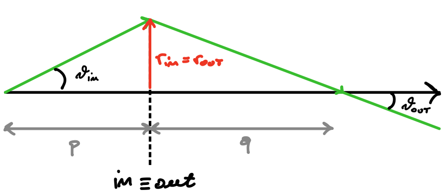
In matrix form we can write:

$$\begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & L\frac{n_0}{n_1} \\ 0 & \frac{n_0}{n_2} \end{bmatrix}}_{M} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix}. \qquad \det(M) = \frac{n_0}{n_2}$$

If $n_2 = n_0$ we obtain the matrix $M = \begin{bmatrix} 1 & L\frac{n_0}{n_1} \\ 0 & 1 \end{bmatrix}$; $|M| = 1$

Positive thin lens

$r_{in} = r_{out}$ and $\vartheta_{in} \neq \vartheta_{out}$. $\quad \frac{1}{p} + \frac{1}{q} = \frac{1}{f}$



$$\begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix}$$

Spherical wavefronts through a lens:

$$R_{out} = \frac{AR_{in} + B}{CR_{in} + D},$$ where R is the radius of curvature.

Inverting the propagation of the rays, we obtain the inverse propagation matrix given by

$$M_i = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}, \quad \text{where} \quad A' = D, \; B' = B, \; C' = C, \; D' = A.$$

If $A = D$, the optical system is called symmetric.

<u>Gaussian beams</u>

Let us consider an electric field transverse distribution in paraxial approximation $\tilde{E}(x, y, z)$. We can approximate the field as

$\tilde{E}(x, y, z) \simeq e^{-jkz} \cdot \tilde{U}(x, y, z)$, where $\tilde{U}$ is a slowly varying envelope containing all the modulations of the field along the transverse plane xy. The main phase changes are due to the e^{-jkz} phase factor.

$\tilde{E}$ must be a solution of the Helmoltz equation

$$(\nabla^2 + k^2)\tilde{E} = 0, \quad \text{where} \quad \nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2}, \quad \nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

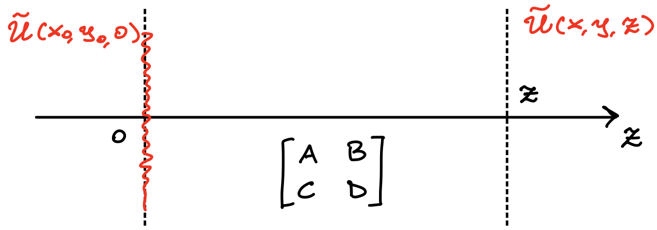
$$\Rightarrow \left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2} + k^2\right)\tilde{E} = \nabla_\perp \tilde{U}(x, y, z) - 2jk \frac{\partial^2 \tilde{U}}{\partial z} + \underset{\text{negligible}}{\cancel{\frac{\partial^2 \tilde{U}}{\partial z}}} = 0$$

Hence, we obtain this new equation, that is the Helmoltz equation in paraxial approximation

$$\left(\nabla_\perp^2 - 2jk\frac{\partial}{\partial z}\right)\tilde{U}(x, y, z) = 0$$

Integrating the former equation for the propagation of the wave through a generic optical system described by its

ABCD matrix in paraxial approximation.



The result is again a mathematical description of the Huygens interpretation

$$\tilde{U}(\vec{r}) = e^{\frac{j}{B\lambda}} \cdot \iint_{\mathbb{R}^2} \tilde{U}(x_0, y_0, 0)\, e^{-\frac{jk}{2B}\left[A(x_0^2+y_0^2)+D(x^2+y^2)-2x_0x-2y_0y\right]} dx_0\, dy_0$$

that is the Sommerfeld integral in paraxial approximation.

We now want to find the eigensolutions of this propagation integral. If they exist it means that, if $\tilde{U}(x, y, z) = \tilde{U}_{in}$, $\tilde{U}_{out} = \tilde{U}(x_0, y_0, 0) \cdot \tilde{a}(z)$, where $\tilde{a}(z)$ is a phase term constant along x and y. We can prove that this solution exists if $\tilde{U}_{in}$ is a Gaussian field distribution, that means:

$$\tilde{U}(x_0, y_0, 0) = U_0\, e^{-jk\frac{x_0^2+y_0^2}{2q_0}}.$$

In this case,

$$\tilde{U}(x, y, z) = \frac{U_0}{A+\frac{B}{q_0}} \exp\left[-jk\,\frac{x^2+y^2}{2q}\right], \quad \text{where } q = \frac{Aq_0+B}{Cq_0+D}$$

$\tilde{U}$ is the Gaussian beam at the output of the ABCD system. Note that the q at the exponential is different w.r.t. the q_0 of the input beam.

A Gaussian beam can be visualized as a spherical wave with a Gaussian transverse modulation of the amplitude.

The electric field of a spherical wave can be written as

$$\tilde{E}_s(x, y, z) = e^{-jkr}, \quad \text{with} \quad r = \sqrt{x^2+y^2+z^2}$$

In paraxial approximation we can write

$$r = z\sqrt{1+\frac{x^2+y^2}{z^2}} \simeq z\left(1+\frac{x^2+y^2}{2z^2}\right) = z + \frac{x^2+y^2}{2z}, \quad \text{hence:}$$

$$\tilde{E}_s(x,y,z) = e^{-jkr} \simeq e^{jkz} \cdot \exp\left[-jk\frac{x^2+y^2}{2z}\right]$$

If z is the propagation distance and the wave origins in $z=0$, we can say that $z=R$, where R is the radius of curvature of the phase front:

$$E_s(x,y,z) = e^{-jkz}\left[\underbrace{e^{-jk\frac{x^2+y^2}{2R}}}_{\tilde{U}_s(x,y,z)}\right]$$

So, the slowly varying envelope for a spherical wave and for a Gaussian wave (only phase terms) can be written, respectively, as:

$$\tilde{U}_s(x,y,z) = e^{-jk\frac{x^2+y^2}{2R}} \quad \text{and} \quad \tilde{U}_g(x,y,z) = e^{-jk\frac{x^2+y^2}{2q}}$$

Thus, a Gaussian beam is a spherical beam with a complex radius of curvature. We can write:

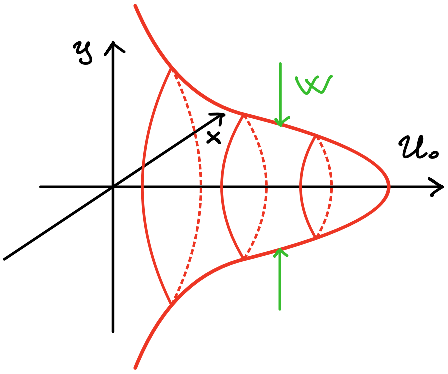
$$\frac{1}{q} = a + jb = \frac{1}{R} - j\frac{\lambda}{\pi w^2},$$ where λ is the wavelength and w is called spot size of the Gaussian beam. The former relation becomes:

$$\tilde{U}_g(x,y,z) = \tilde{U}_0 \cdot \exp\left[-\frac{x^2+y^2}{w^2}\right]\exp\left[-jk\frac{x^2+y^2}{2R}\right]$$

Thus, we have a spherical phase distribution times a real Gaussian envelope that modulates the envelope of the field.

A graphical interpretation of the concept is shown below.

The electric field of a Gaussian beam is given by:



$$\tilde{E}_g = e^{-jkz} \exp\left[-jk\frac{x^2+y^2}{2R}\right] \exp\left[-\frac{x^2+y^2}{w^2}\right]$$

The maxima of the phase are obtained if:

$$\varphi(x,y,z) = k\left(z + \frac{x^2+y^2}{2R}\right) = 2m\pi$$

So, we have the surfaces of maximum phase for

$$z = -\frac{x^2+y^2}{2R} + m\lambda \quad \text{(rotation paraboloids around the } z \text{ axis)}$$

Let us assume that the beam propagates in free space. The propagation matrix would be $\begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \Rightarrow q(z) = \dfrac{q_0 + z}{0+1} = q_0 + z$

Let us consider a boundary condition for the beam in $z=0$:

$$\frac{1}{q_0} = \frac{1}{R(0)} - j\frac{\lambda}{\pi w_0^2}, \quad \text{with } R(0) \to \infty \text{ (plane wave)}$$

In this way q_0 is purely imaginary: $q_0 = j\dfrac{\pi w_0^2}{\lambda} = j z_R$,

where z_R is called Rayleigh distance of the beam.

Hence, $$\frac{1}{q(z)} = \frac{1}{q_0+z} = \frac{1}{jz_R+z}\cdot\frac{z-jz_R}{z-jz_R} = \frac{z}{z^2+z_R^2} - j\frac{z_R}{z^2+z_R^2} = \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

$$R(z) = z\left[1 + \left(\frac{z_R}{z}\right)^2\right]; \quad w(z) = w_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}$$

Let us focus on $R(z)$. $\lim_{z\to\infty} R(z) = +\infty$. Since it is continuous, the function $R(z)$ must have a minimum. We have the minimum for $z = z_R \Rightarrow R_{MIN} = R(z_R)$.

Hence, if $z \ll z_R$, R is big and the phase fronts can be considered plane. If $z \gg z_R$, $\frac{z}{z_R} \to \infty$, hence $R(z) = z$ and the phase fronts can be considered spherical.

Let us now study the propagation of the spot size.

$$W(z) = W_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}.$$

$z \gg z_R \Rightarrow W(z) = W_0 \frac{z}{z_R}$

$z \ll z_R \Rightarrow W(z) = W_0$

Calling ϑ_d the divergence angle, we can write

$$\vartheta_d = \tan \vartheta_d = \lim_{z \to \infty} \frac{W(z)}{z}$$